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ELECTRODYNAMICS OF MOVING MEDIA

Report of the National Research Council Committee on Electrodynamics of Moving Media

By

W. F. G. Swann, John T. Tate, H. Bateman, E. H. Kennard

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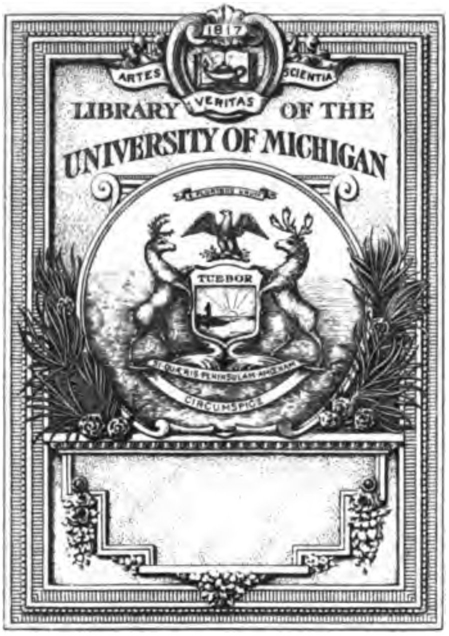

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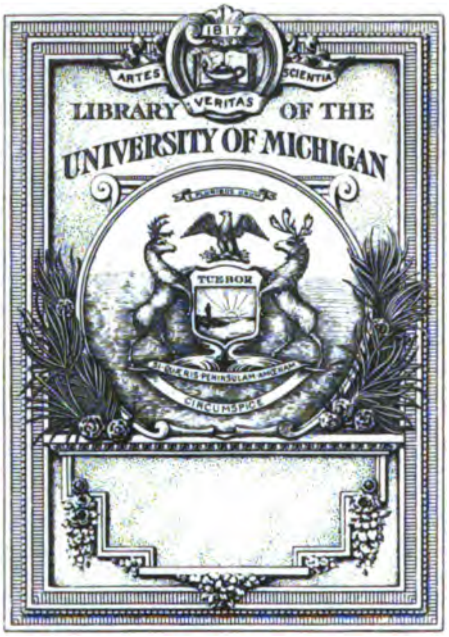

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ELECTRODYNAMICS OF MOVING MEDIA

Report of the National Research Council Committee on Electrodynamics of Moving Media*

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PART II

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PART I

THE FUNDAMENTALS OF ELECTRODYNAMICS

BY W. F. G. SWANN

INTRODUCTION

The present report purposes to make a critical survey of some of the more familiar aspects of the subject with a view of inquiring as to what parts may be considered as relying upon experiment, or upon deduction from fundamental principles, and what parts are mere definition or scaffolding, convenient for the correlation of the experimental laws but in no sense calling for experimental proof themselves. Any individual experiment usually covers a very limited region of the whole field of investigation of which it forms a part. It is a representative of a much larger class, from which it is chosen on the grounds of practicability. In citing an experimental foundation in any particular case we shall not therefore confine ourselves to individual experiments which have been performed, but shall speak rather of general classes of experiment of which any particular experiment will form a special case.

One method of procedure would be to take the complete scheme of equations as formulated by Lorentz, and discuss everything in terms of these. To do this, however, would be to clothe the subject with considerable artificiality as regards certain important fields of its application. The attitude which one adopts toward the subject, his way of thinking about it, even the method of expression of its laws is to some extent bound up with the degree of generality which he has in mind for its application. Thus, for example, there is a perfectly consistent scheme of hypotheses applicable to the case where one confines himself to unvarying currents in closed wire circuits; and while this subject may be treated as part of the general case of electronic motion, it may as far as its own requirements are concerned, be discussed in different and somewhat simpler language. The field of application of this special case is moreover so great that it constitutes as it were a little subject in itself, and it is therefore convenient to discuss what would be its laws from the standpoint of one who never had occasion to deal with more general cases. Following out this plan, we shall consider the whole subject in four stages applicable respectively to the following cases.

1. Electrostatics.
2. Steady currents in closed circuits.
3. Varying currents in closed circuits.
4. The general case as symbolized by the Lorentzian equations for free aether and their adaptation to a material medium.

Certain matters of detail will be relegated to notes in an appendix to this report.

ELECTROSTATICS

The laws of inverse squares and of conservation of charge.—The fundamental law is here, of course, the law of inverse squares; but, the meaning of this law is to some extent a function of the light in which we regard it. We may speak of it in terms of the law of force between certain material bodies, and we may in fact formulate the general law in the following way:

> Let there be given an assemblage of bodies. Then it is possible to assign to each element of volume *of the bodies* a definite number ρ such that on writing down vectors
>
> $$\rho_1\rho_2 d\tau_1 d\tau_2/4\pi r^2$$
>
> for the forces between each pair of corresponding elements, the forces so obtained when compounded will give the resultant mechanical forces and couples exerted by the bodies on each other.
>
> Moreover, if the bodies be moved to new positions, a similar result holds, and the values of $\int\rho d\tau$ for any one of the bodies in the two positions are equal.
>
> A

Surface distributions may, of course, be included as limiting cases of volume distributions.

This is the law which may be regarded as the experimental basis of electrostatics. In this form the definition of charge density is not made until the law has been recognized, and indeed until then it has no meaning. The superposability of effects, the proportionality of mechanical force to charge, and the conservation of charge are contained in the statement of the law, once the experimental fact has been assumed.

A consideration of the matter will show that all electrostatic experiments designed to test the law of inverse squares are particular cases of a general experiment of this type. An experiment with Coulomb's torsion balance is of course of this type; and while the statement may not be obvious at first sight, a little consideration will show that even experiments having to do with the absence of field inside a hollow conductor are of this form. Thus in its most general form this experiment with the hollow conductor may be taken to show that a charged body placed inside such a shield experiences no mechanical force as a result of the presence of charges outside. Now it is a mathematical fact that whatever distribution ρ be assigned to the space outside of a surface, it is always possible to assign a function σ varying over the surface in such a way that while $\int\sigma dS = 0$, the vectorial sum of

$$\iiint\frac{\rho d\tau}{r^2} \quad \text{and} \quad \iint\frac{\sigma dS}{r^2}$$

is zero for any point which is inside the surface, and whose distances from the various volume elements $d\tau$ and surface element dS are given by the corresponding values of r. It follows that if the mechanical force upon a body is given by a law of the type above specified, it is certainly possible to assign a distribution σ which when combined with the distribution ρ will result in zero mechanical force on a charged body within the surface, and in the limit on the surface. The fact that there are material surfaces (conductors), whose nature is of a kind such that the distribution σ will automatically come about is something which is outside the scope of assertion of the fundamental law. Once experiment has revealed that there *are* surfaces inside which a body will experience no force although bodies outside that surface attract and repel each other, we have realized a situation which certainly can be accounted for by a law of the type *A* preceding, and it is possible to prove with a fairly large degree of generality that a law of this type is the only one possible to account for the phenomenon.

In the above formulation of the law, no mention of a field is made It is, however, a mathematical convenience to *define* now a quantity *E* as

$$4\pi E = Grad \iiint \frac{\rho}{r} d\tau. \tag{1}$$

This definition involves nothing about the force on a unit charge, and it is automatically contained in the experimental law as above stated that the force on an element of matter charged with density ρ is $\rho E d\tau$.

It may be parenthetically remarked that the possibility of relegating *E* to a minor role as regards the expression of the law is not one which is confined to the statical case, but can be employed with advantage even in the most general case, and is indeed the only logical way of formulating the laws in a manner capable of experimental proof. Thus, for example, nobody can assign any meaning to the field inside a moving electron when that field is defined as the force on a unit of charge attached to a piece of matter of macroscopic size. In the light of the above statement, one might question the necessity for introducing such a quantity as *E* at all. It is true that *E* need play no fundamental role as regards the expression of the fundamental laws; but, there are a number of facts, concerned with the special properties of materials, which are of enormous importance from the practical standpoint although they are not to be regarded in the light of fundamental laws. Such a fact is the existence of a body possessing the properties of what we call a conductor. A conductor is a body in which the charge distribution must adjust itself in such a way that the quantity *E* as above defined will always be zero within its substance. It is thus in terms of this subsidiary quantity *E* that this non-fundamental but very valuable property of a conductor becomes expressed.

In spite of what has been written above concerning the advantages to be derived from relegating *E* to a subsidiary role in the matter of expression of general laws, one must face the fact that it is customary to speak of a quantity E which is regarded as defined in terms of the usual unit charge, a convenient silence being maintained as regards the realization of this definition in such a case as that of a point inside an electron for example. Without therefore attempting to justify this attitude, or even to remove the logical difficulty in respect of such an imperfect definition, it is of interest to consider what becomes of the law of inverse squares for a case where our starting point is the hypothesis that E is defined or is capable of definition at any point in space.

Case where electric field is the quantity fundamentally defined.—The first thing that is necessary is a definition of charge density. Charge per unit volume would only move matters a step farther back, by calling for a definition of charge. Without belaboring this point unduly we may recall that the necessity of making a definition of ρ which is applicable in all cases, including that of a moving charge, has resulted in its being defined in terms of the field as

$$\rho = Div\ E \tag{2}$$

With this definition as applied to the statical case, however, a rather remarkable result follows.

It is a result of pure mathematics, and independent of experiment, that any vector E can be expressed at a point P in the form

$$4\pi E = -Grad \iiint \frac{Div\ E}{r} d\tau + Curl \iiint \frac{Curl\ E}{r} d\tau \tag{3}$$

where the volume integrals are taken throughout all space, and r is the distance of an element of volume from the point P.

If we make the assumption that the vector E has no curl, and we may take this as our definition of what we mean by limiting ourselves to an electrostatic system, we have

$$4\pi E = -Grad \iiint \frac{Div\ E}{r} d\tau \tag{4}$$

If ρ is defined as $Div\ E$, this is nothing more nor less than the expression of the law of inverse squares for the entity whose density is given by ρ. If, for example, we have a point 0, and we assign a field which varies inversely as the cube of the distance from 0, it is obvious that the field in question will give rise to definite values of $Div\ E$ throughout space, and the above mathematical theorem shows that the amount of this is just such that the field may be regarded as one of the inverse square type, produced however by a distribution of charge which is not confined to the point 0.[1] It is useless to say that there is no charge save at 0, for in terms of *Div E* as the definition of ρ, the very act of assigning such a field as one varying according to the inverse cube will automatically carry with it the assignment of the charge distribution necessary to represent this field as consistent with an inverse square law. These remarks are particularly pertinent to such cases as those where it is customary to speak of the forces between electrical charges in atoms as obeying laws of force other than the inverse square, even in the statical case. Such laws need be regarded as in no sense inconsistent with the classical theory, and indeed cannot be.

---

[1] The infinite charge which will be necessitated at 0 may be avoided by specifying the inverse cube law as applying only outside of a sphere of finite radius.

What then remains of the experimental proof of the law of inverse squares for the type of formulation which defines ρ as *Div E*? We must ask what it is that we wish to prove. The most reasonable answer is that we wish to prove that there are certain conditions in which the field in a given region can be expressed entirely in terms of fields due to charges outside of the region, which charges act according to the inverse square law. From the standpoint we have adopted, however, this result will follow as a consequence of pure mathematics provided that we have *Div E* zero throughout the region. The whole experimental law can thus be formulated in the statement that there are regions in which *Div E* is zero.

Our search for a region in which *Div E* is zero must of course be made in the light of a precise definition of *E*. As regards phenomena on a sufficiently large scale the usual definition in terms of unit charge will suffice. We now observe that if we can find a case in which *E* is zero throughout a region, (although not of course zero everywhere outside, for this would give a trivial result), we shall have found a region in which *Div E* is zero. Now experiment shows that there are regions, hollow conductors, in which we can have *E* zero while it is not zero outside; and from the standpoint which we are now considering, it would appear that all that this experiment of the hollow conductor may be taken to establish is that there are regions in which *Div E* is zero while it is not zero outside. We have apparently not made much use of the fact of the boundary of the region's being a conductor. Indeed the only essential thing is the *existence* of a region in which *E* and consequently *Div E* is zero. It is however possible to go on and show that if such a region exists, its boundary must be an equipotential surface, although this is not a fact of which the proof makes direct use.

Although the definition $\rho = \mathit{Div}\ E$ would carry with it the law of inverse squares for the field in the foregoing sense, an extra hypothesis to the effect that the mechanical force on an element of matter is proportional to the charge on it is necessary; for, the definition of *E* as the force on a particle of matter containing unit charge does not carry with it as a necessary consequence, the proportionality of the force on the matter and the charge when the latter is defined in terms of a density given by *Div E*. The assumption of the said proportionality would therefore be one subject to the necessity of experimental investigation. This experimental fact, combined with the assumption, which must be regarded as based upon experiment, that certain types of charge distribution exist, that in fact charge is only to be found in the places where we choose to specify it in any given problem, these are the only facts relying upon experiment for a field whose curl is everywhere zero, and in which the charge density ρ is defined as *Div E*. The law of inverse squares is for this case a result of pure mathematics.[1]

Dielectrics.—As regards the role played by dielectric phenomena in electrostatics little is to be said. The fundamental assumption is that the dielectric may be represented as regards its external action by replacing it by a distribution of polarization P. This turns out to be the mathematical equivalent of a fictitious volume distribution of charge with density $-Div\ P$, and a fictitious surface distribution which at each point of the surface has a density equal to P_n, where the subscript n is to be taken as denoting the component in the direction of the outward normal. Thus, in order to include dielectrics, no further extension is needed in the experimental law as formulated above with the electric field at an external point occupying the position of a subordinate quantity introduced after the truth of the law has been recognized. As regards the field at a point inside the dielectric, this is defined as the quantity obtained by drawing a small cavity about the point, measuring the field therein, and subtracting the portion of the field due to the fictitious charge on the wall of the cavity, this latter being the only part of the field within the cavity which does not attain a definite limit as the size of the cavity is made infinitesimal. We may parenthetically remark that, although it is not customary to do so, we may avoid completely the mention of a cavity by defining the field at a point within the dielectric as the actual field which would be measured at that point if we were to remove the dielectric, and replace it by its fictitious volume and surface charges. The smoothing out process implied in the specification of the fictitious charges carries with it the requirements necessary for insuring a definite limiting value for the field at the point, without resort to the use of a cavity. This method suffers in no respect in comparison with that invoking the use of a cavity in that it requires specification of the fictitious charges before the field can be obtained;

---

[1] As regards conservation of charge, perfect generality of view-point would hardly call for a more restrictive hypothesis than the assumption of a constant and zero value for the integral of *Div E* for the universe as a whole. Denial of the possibility of coalescence of the ultimate positive and negative units would permit of a more restrictive hypothesis in the form that the volume integrals of *Div E* taken separately for positive and negative values of *Div E*, and in a true differential as distinct from a macroscopic sense, are each individually constant for all time, and are equal, though opposite in sign.

for, although it would be possible to measure the field inside a cavity without invoking the knowledge of the fictitious charges, we should require to know these charges in order to calculate the contribution of that portion of them associated with the wall of the cavity. There is one point in favor of the cavity-definition however. We may construct a long narrow cavity, and can conceivably turn it about within the medium until we find experimentally that the field inside it is parallel to its sides. We may then define the field at a point in the dielectric (if non-crystalline) as the field at a point so chosen within this cavity that the distance of the point from the ends is large compared with the width of the cavity. Such a definition is independent of any theory of fictitious charges or of doublet action of the dielectric. Until we assume some such theory, however, we are unable to attach any properties to the said field. It derives its properties only through the particular theory of the mechanism of the process which has suggested its definition. The most important property of E is that $E+P$ is a solenoidal vector in places where the true macroscopic charge density is zero. The introduction of this vector $D=E+P$ is a mathematical device designed to relieve the mind of the necessity of thinking of the fictitious charges and to allow it to concentrate attention only upon that portion of the total charge which is not directly attributable to the doublets, and whose amount is measured by *Div* D.

The relation between D and E is a matter of experiment, but it is something which does not partake of the nature of a general law. It is a happy accident that the ratio D/E is a constant in most substances, but it is not a result which may be dignified by being recognized as a fundamental law of physics.

Regarding the boundary conditions at the surface of dielectrics, the continuity of the normal induction is a result following directly from the definition of that quantity; and while the equality of the external and internal tangential intensity is frequently proved by appeals to the conservation of energy and founded on the carrying of a unit charge around a circuit partly in one medium and partly in the other, a close examination of the matter will show that this demonstration assumes the result to be proved, and that the proof itself rests upon the definite assumption as to how the dielectric acts in being the equivalent of fictitious volume and surface charges. This assumption carries with it the equality of the tangential components in question. Any appeal to the conservation of energy as an experimental basis requires the assumption of continuity of potential at the boundary, or at any rate a discontinuity which remains constant to the second order over a first order distance parallel to the boundary. When this assumption is made, however, the equality of the tangential components follows immediately without any appeal to the conservation of energy.

Steady Currents in Continuous Circuits

The unit pole is an obviously obnoxious concept for the purposes of a fundamental definition since it does not exist. The objection may be in part removed by utilizing a unit magnet and defining the field in terms of the couple on it. Even here, a great deal of care is necessary in preserving logical exactness in the definition of magnetic field. (See appendix, Note 1). Two laws are customarily associated with the action of steady currents. The first of these concerns the field due to a current, while the second concerns the forces on a conductor carrying a current. The first law may be expressed in either of two equivalent forms;

> **(a) That the field due to a current circuit is the equivalent of that of a magnetic shell of uniform strength whose boundary is coincident with the circuit.**
>
> **(b) That a current circuit acts as though each element produces a field at a point *P* in a direction perpendicular to the plane containing the element and radius vector joining it to *P*, inversely proportional to the square of the distance, and directly proportional to the resolved component of the element perpendicular to the radius vector, and to the current.**

B

We have a situation analogous to that met with in the casė of electrostatics. If we regard currents as things produced in certain specified ways, and if when we have a number of visible wire circuits we decide to say that what we shall call current exists only in those circuits, and that there are no invisible current circuits in space, then the above law must be regarded as an experimental fact. If, however, anticipating the future needs applicable to the case where we have to deal with invisible movements of electricity, anchor rings of moving electricity in atoms, etc., we define the current density as *curl H*, then remembering that any vector *H* may be expressed as

$$4\pi H = -Grad\int\int\int\frac{Div\,H}{r}\,d\tau + Curl\int\int\int\frac{Curl\,H}{r}\,d\tau \qquad (5)$$

we see that nothing more than the assumption, or if we like the experimental criterion, that *Div H* is zero is necessary in order that our definition of the current density *j* as *curl H* shall lead to

$$4\pi H = Curl\int\int\int\frac{j}{r}\,d\tau \qquad (6)$$

which, by simple analytical procedure turns out to be the law of the magnetic shell in the case of a circuit.

Just as in the corresponding case in electrostatics, with ρ defined as *Div E*, the complete role of experiment was played in saying that certain types of charge distribution existed, that in the case of a certain set of bodies there was no charge save on the bodies, so here the complete role of experiment is played in assuring us that when we do to wire circuits those things which we associate with the production of currents in them, the currents will exist in those wires and nowhere else.

The second law concerned with the action of continuous currents has to do with the action of a magnetic field upon a current, and it is customary to derive it as a consequence of a rather unsatisfactory procedure founded upon the consideration of the force due to the current on a pole, and the assignment of forces on the current due to the pole in such a way that the two shall be in statical equilibrium. The force on the current element is then expressed in terms of the field of the pole and the result is extended to apply to the general type of field which may be considered as made up of the field due to a suitable assemblage of poles.

When all is said and done, the law as finally expressed takes the form that,

> **The forces and couples on the circuit as a whole are the same as would be secured were we to assign to each element ds, of the current circuit, a force of the form**
>
> $$\Delta F = ds[i.H] \tag{7}$$

Now just as in the electrostatical case the introduction of the field E, with ρ defined as *Div* E, necessitated the assumption of a law as to the types of charge distribution which are possible, and then another law stating how the field acts upon the charged bodies, while unless we are interested in the value of the field for its own sake, its introduction is unnecessary, so here, unless we are interested as a matter of pure philosophical speculation in the question of the magnetic field itself, we may relegate it to a subsidiary role, formulate the law which is to be considered as the result of experiment without it, and then introduce it again, defined in a new way, however, for the purposes of mathematical convenience.

We may in fact state in the following way the complete law of the action for continuous steady currents, i. e., the only law which calls for experimental investigation.

> **Suppose we have a number of material current circuits. Then it is possible to assign to the circuits numbers i_1, i_2, i_3, etc., such that if H be *defined* as**
>
> $$H = \Sigma \mathit{curl} \int \frac{i}{r}\, ds \tag{8}$$
>
> **the forces and couples between the current circuits are given by assigning to each element a force**
>
> $$\Delta F = [i.H]\, ds \tag{9}$$
>
> **where at each point of the nth circuit, i is to be regarded as a vector of modulus i_n. Electrostatic forces are supposed eliminated by screens, which prevent no difficulties for steady currents.**

C

Here H is retained merely as a symbol for mathematical convenience. It involves nothing about a pole or a magnet. Indeed, we could dispense with it altogether by expressing our force on the element ds_n as,

$$\Delta F_n = \left[i_n \Sigma curl \int \frac{i}{r} ds \right] ds_n \tag{10}$$

It will be observed that there is a certain analogy in language between the expression of this law and the fundamental law of electrostatics as given on page 6. In the present case we have omitted the part which would correspond to the requirements of constancy of total charge in electrostatics. Were we to incorporate this statement in the present case it would necessitate the addition to C of a statement to the effect that,

> If the circuits be then moved to other positions a similar law holds, and the new values of the i's for the several circuits will be the same as the old values. } D

The objection to this addition is that, in the ideal and general case, it would not be true, as will immediately be surmised if one thinks of the permanent alteration in current which would be produced in a resistanceless circuit by electromagnetic induction if its position were changed in such a way as to alter the magnetic flux through it. In all practical cases, however, where we have resistance, we know that such effects as those cited would be only transitory, the currents produced by electromagnetic induction dying to negligible values in sufficient time, leaving only the contributions to the currents by those other sources, batteries, etc., with which we are principally concerned in speaking of steady currents. If, however, we wish to retain a degree of formality which is perhaps out of proportion to its practical usefulness we should limit ourselves to the statement that "there are cases in which the law stated in C with the addition D is true."

It must be pointed out that, even though we restrict ourselves to the form C of the law without the addition of D, the law itself forms more than a mere definition of the current strength. For while, with n given circuits, we could write down n equations for the force components on the circuits in any direction, and determine values of i so as to satisfy these relations, independently of the form of the law which had been assumed, and while nothing more than the vectorial form of the relations would be necessary in order that these same i's would lead to the proper components of the force in the perpendicular directions, it would not necessarily follow that the couples would be correctly given by the same i's as gave the proper forces. It may be argued that, in the practical operation of testing the law of action of currents, we do not go through the artificial process of assigning numbers i_1, i_2, etc., but adopt processes which are much more in line with our physical conceptions. A little thought will show that if we remove from our minds all pictures concerned with the visualization of the process, and concentrate only upon what we actually do, our tests do really take this form. Thus, to take the simplest possible case, that of two circuits with constant currents, were we to set out to test our law, we should write down an equation founded upon equation (10), for the couple on one of the circuits and we should calculate two numbers i_1 and i_2, one for each circuit. We should then move the circuits to different relative positions and test whether the same two numbers i_1 and i_2 would serve in conjunction with equation (10) to give the couple in this case.

Although in the form of expression of the general law as symbolized by (10) the magnetic field has been dispensed with, it is naturally a convenience for mathematical manipulation to reintroduce it defined by equation (8). The properties of H are now of course all contained in its definition, and this definition gives to it the property of being expressible in terms of magnetic shells as before. Even the relation $Div\ H=0$ is secured by the relation (8), since the divergence of a "curl" is necessarily zero.

In turning things around in this way and formulating everything in terms of one experimental statement we may have a suspicion that we have lost something, and indeed we have, but happily something of very little value. In terms of the present specification there is nothing to say how that specialized piece of steel we call a magnet will behave under the influence of the currents and the field H as now defined. If we wish to bring the steel magnet in as a matter of historical interest, we may do so by adding an additional experimental conclusion, viz., the conclusion that the said piece of steel can be treated as regards its behavior as though made up of current whirls.

In concluding these statements regarding steady currents we may remark that in the formulation and definition of the field H in terms of the currents, and of the latter in terms of the forces between the circuits, we formulate our laws according to the actual principles which are used in accurately measuring the currents and fields.

It must be added that the proportionality between the currents as above defined and the currents as defined by electrolysis is to be regarded as an extra fact which must seek its foundation in experiment.

VARYING CURRENTS IN CLOSED CIRCUITS

Dynamical systems.—Since so much of the formulation of the general laws of electromagnetism is bound up with dynamical methods, the deduction of laws from dynamical principles and so forth, one has to consider to what extent these processes constitute proofs of the results deduced, and in what place experiment plays its part. It will be well, therefore, to make a few remarks about dynamical systems and developments in general. First, what is a dynamical system? This is of course a matter of definition. There are several definitions, but the one on which it will serve to fix attention is this:

Suppose it is possible to define a set of measurable quantities θ_1, θ_2, θ_3, etc., and to build up a homogeneous quadratic function T of the time rates of change of these quantities, and another function V of the quantities themselves such that the motions of the system are found to be in agreement with the equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} = -\frac{\partial V}{\partial \theta_1};\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} = -\frac{\partial V}{\partial \theta_2};\text{etc.,} \tag{11}$$

then the system is said to be a dynamical system. If it be more convenient, we may use in place of (11), their analytical equivalent as specified by the Hamiltonian principle.

$$\delta\int(T-V)\,dt = 0 \tag{12}$$

It is occasionally necessary to subject the coordinates to constraints, with the introduction of undetermined multipliers in a perfectly well known manner which we need not belabor.

The only birth-mark of this rather artificial specification of a dynamical system which gives it, in its general form, some connection with those particular systems of particles which are discussed in text books in which Lagrange's equations are deduced, the only birth-mark which relates it to what, in our youth, we regarded as familiar, is a property of such a system which was discussed by Poincaré, to the effect that whatever quantity T we assign of the above type, and whatever quantity V we assign of the above type, it is always possible to assign a system of particles with constant m's for which $\Sigma\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ is this function T, and to assign central forces between the particles, and on them from without, such that, when we go through the process of building up Lagrange's equations for these particles, our V will turn out to be the quantity whose derivatives are these external forces; in fact, it is always possible to assign a system of particles controlled by central forces such that the Lagrangian equations which we have are the generalized equations for these particles. Moreover, this can be done in an infinite number of ways. We must not always expect that the particles to which the dynamical system is to be referred in this way are necessarily such particles as electrons for example. There is danger that when we meet things like electrons, we shall at once assume that these are the fundamental particles in terms of which the system is to be expressed if it is to figure as a dynamical system. There is danger of our being led, as some have been led, to the conclusion that, because we cannot express the behavior of the electromagnetic field in a dynamical form with part of the kinetic energy T given by $\Sigma\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ for the electrons, therefore the system is not expressible in dynamical form, whereas the very nature of the dynamical development so expresses it. There is indeed no earthly reason why the ultimate particles to which the system might be referred should have any simple relation whatever to such particles, electrons, and so forth, as we happen to find flying about.

With regard again to the role of theory versus experiment, it must be remembered that the function of a dynamical development in electrodynamics is not to prove anything. The situation is not one where we ascertain in some mysterious way, as a result of crude experimental illustration or otherwise, that certain expressions for T and V are the right ones and then go ahead and prove something which calls for no further experimental investigation if the laws of dynamics are true. The situation is rather the reverse. The things to be experimentally verified are the results, and what has been done is to mould the whole scheme into a dynamical form by finding what expressions for T and V must be chosen in order that the application of the equations of Lagrange shall lead to these experimentally established results. The fact that the expressions chosen for T and V fill this role is the only criterion for their truth. Any one who adopts any other view, and tries, for example, to argue that $\frac{1}{2}E^2$ represents the potential energy per unit volume in the general case, because this expression was deduced in the case of electrostatics, will find a very confusing piece of work on his hands if he attempts to carry out, for the general case, any argument analogous to that which he employed as a proof in the statical case.

In Maxwell's original investigation applicable to continuous circuits, the starting point was the assumption that the kinetic energy was a sum of two parts, one a material part having exactly the form applicable in the absence of currents and another part in which the velocities were represented by the currents, this part being in fact a homogeneous quadratic function of the currents of the form

$$T=\tfrac{1}{2}L_{11}\,i_1^2+L_{12}i_1i_2+\;.\;.\;.\;L_{mn}\,i_mi_n\;.\;.\;. \tag{13}$$

with the L's functions only of the material coordinates. The absence of product terms of material and electrical velocities was inferred from the fact that the presence of terms of this type in the energy would cause Lagrange's equations to predict phenomena which were found experimentally not to exist. Terms of this kind would, for example, lead to the conclusion that there would be mechanical forces on the conductors depending upon the rate of change of the currents, whereas experiment shows that all such forces depend only on the currents themselves. Then again there would arise mechanical forces on conductors of amounts proportional to the product of the mechanical velocities and the currents, forces which were again sought without success. Again there would be electromotive forces as a result of mere acceleration of the conductors even though initially they had no currents in them. It may be remarked that, from the standpoint of electron theory these effects sought for by Maxwell should actually exist. They should be just within the range of measurement, and indeed, some of them have been observed, although of course they were quite beyond the means of detection at Maxwell's disposal.

The utilization of the foregoing energy expressions in the Lagrangian equations leads immediately to the following two expressions for the generalized mechanical force X on a circuit denoted by subscript (1), and the generalized electrical force Y operating on that circuit. These generalized forces are defined as the coefficients of the displacement of the corresponding coordinate in the expression for the virtual work.

$$X_1 = i_2 i_1 \frac{dL_{12}}{dx_1} + i_3 i_1 \frac{dL_{13}}{dx_1} + \dots \qquad (14)$$

$$Y_1 = \frac{d}{dt}\left(L_{11} i_1 + L_{12} i_2 + L_{13} i_3 + \dots\dots\right) \qquad (15)$$

These then are the expressions which it remains for experiment to test if the above expression for the energy is to be regarded as a possible one to figure in a dynamical specification of the laws.

It is a matter of pure analysis to show that provided that L_{mn} be chosen as the purely geometrical function of the shape and positions of the circuits given by

$$L_{mn} = \int\int \frac{\cos\varphi \, ds_1 ds_2}{r} \qquad (16)$$

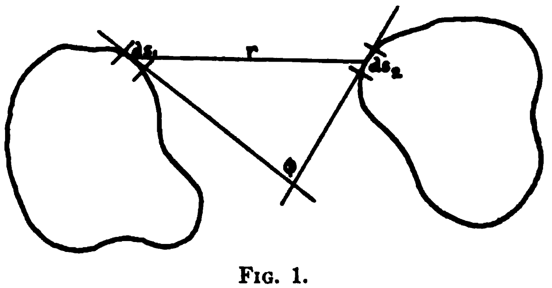

FIG. 1.

expression (14) is the analytical equivalent of equation (10) which we decided to regard as the experimental foundation for the laws for steady currents. The correlation for the case of steady currents fixes the L's to have the values given by (16), but once this is done, the expression

(14) reverts to that form for steady currents, and retains this form as regards what it has to say about variable currents.

Moreover, it follows as a pure analytical consequence of the choice of the above expression for the L's that the expression in parentheses in equation (15) represents the flux through the circuit (1) of the quantity H already defined by (8), so that (15) bears a close resemblance to Faraday's law.

As regards the mechanical action of the current circuits on each other, the law to be tested is then the same as the law for steady currents. The question is whether it is possible to chose numbers i_1, i_2, etc., one for each circuit, such that the law of mechanical action may be expressed in terms of them in the manner we have specified, after inclusion of appropriate electrostatic forces, since shielding is not now permissible. Although it would be very difficult to show by ordinary laboratory experiments that such was impossible, we may draw upon our further knowledge of the laws for open circuits to convince ourselves that were we able to carry out the experiment with sufficient precision, it would be found impossible to choose numbers i_1, i_2, etc., fulfilling this condition, even by inclusion of electrostatic forces. The defect of the theory is that it ignores the finite time of propagation of effects. In the open circuit theory where, for the first time, the currents of which we have spoken become definitely associated with electric charges, it becomes necessary in order to retain certain properties which are at the basis of our physical concepts of the subject, to introduce the displacement current density $\dot{E}$, which, of course, immediately introduces the analytical characteristics of a finite rate of propogation of effects. Continuous current theory which had not yet recognized any necessary connections between the currents as defined and the motions of electric charges would not be impelled on its own account by any strong logical call for the existence of this displacement current. Once $\dot{E}$ has, however, been introduced in the more general case it cannot be summarily discarded for the more special case unless it disappears of its own accord, which it does not do.

We may infer from this that although it is the more general case for open circuits which has suggested the introduction of the displacement current, the field of closed currents would not, at any rate from an ideal standpoint, have to rely upon open circuits to show the incorrectness of the theory we have discussed, but would, of its own accord, reveal this incorrectness in the law of mechanical action of the circuits if only the experiments could be carried out with sufficient refinement.

As regards equation (15) giving the electromotive force, this would be Faraday's law provided that a way could be found of associating Y with the electromotive force in the sense that that term is used in Faraday's law. It is important to realize that, at this stage no suggestion has been made as to Y_1's having anything to do with electricity. It is defined as the coefficient of the current i_1 in the expression for the rate of doing work by the external forces. As such, the nearest we could come to its experimental realization would be to measure it as the coefficient of the current i_1 in the expression for the rate of generation of heat in the circuit (1), a procedure which would, however, in order to maintain logical precision, necessitate the introduction of a number of other postulates and assumptions, or experimental facts, having but a limited bearing upon the matter in hand.

We might take (15) as the definition of electromotive force, and then define resistance as the ratio of the e.m.f. so defined to the current in the conductor (1). The sole fact which would then come out of experiment would be that there exist conductors for which the resistance so defined is independent of the current (i. e., Ohm's law holds). This is really all that is proved by an experiment in which the throw of a ballistic galvanometer is measured as the result of starting a current in some other circuit, for Ohm's law is assumed in the interpretation of the result in relation to what it is desired to prove.

One might say, why not introduce physical reality into the quantity Y, and at the same time tie it up with electrostatic phenomena by introducing a battery,[1] which is a contrivance having a very obvious connection with electrostatics, in view of the fact that its plates are at different potentials as tested electrostatically. The trouble is that a battery is not a thing which it is very easy to define mathematically. It is a sort of electrostatic device which produces a closed current; and to account for this astonishing performance, we have to introduce a lot of extraneous experimental and theoretical material, such as Nernst solution pressures, etc., with which it is undesirable to encumber the argument for the sake of forcing the connection between currents and electrostatics into the discussion at this stage, particularly as the connection becomes more naturally discussed in the more general analysis for open circuits, and as the whole theory in the form in which we are discussing it is experimentally wrong, since it neglects the finite time of propagation of effects.

One might say, why discuss a theory which is wrong? The answer is, that for many practical purposes it is sufficiently nearly right, and is not in itself logically inconsistent in its individual parts. All that it is useful to assert, as regards the Faraday law, at this stage of the development, is that equation (15) is not to be regarded as an experimental law, but as the definition of e.m.f. The ratio of the so defined e.m.f. to the current in the wire is called the resistance in electromagnetic measure, and the usefulness of this definition lies in the fact that, for many substances, if the changes in current are not too rapid, the so defined resistance is a constant independent of the magnitude of the current. Connections with electrostatics can only be brought in with the accompaniment of extraneous and cumbersome considerations involving other matters, and are better relegated to the more general part of the subject concerned with open circuits.

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[1] A condenser is not allowable in the present stage of our discussion, since we are here confining ourselves to the laws of closed circuits.

Without incorporating the statement as part of any general law, however, it may be convenient to recognize, as an experimental fact, that the quantity defined as the electromotive force in (15) is approximately proportional at any instant to the open circuit difference of electrostatic potential between the poles of a battery which, when forming part of the circuit, would at that instant result in zero current.

THE GENERAL EQUATIONS

The equations are usually written:

$$\frac{1}{c}\left(\rho v+\frac{\partial E}{\partial t}\right)=Curl\ H \tag{17}$$

$$-\frac{1}{c}\frac{\partial H}{\partial t}=Curl\ E \tag{18}$$

$$Div\ E=\rho \tag{19}$$

$$Div\ H=0 \tag{20}$$

and to these we must add the force equation, which states that,

"The motion of an element of charge (or in the form in which the law is used, an electron) is determined by the condition that the value of

$$\rho\{E+[vH]/c\} \tag{21}$$

integrated over it is zero."

It will perhaps conduce to clarity if we pause for a moment to consider the manner in which these equations are related to each other as part of a dynamical system.

The expression

$$T=\tfrac{1}{2}L_{11}\,i_1^2+L_{12}\,i_1i_2\quad+\text{etc.} \tag{22}$$

taken by Maxwell for the kinetic energy of a system of current circuits becomes generalized into the form

$$T=\tfrac{1}{2}\int\int\int(J_xj_x+J_yj_y+J_zj_z)d\tau \tag{23}$$

when applied to continuous distributions of current with density *j*. In this expression,

$$4\pi J=\int\int\int\frac{j}{r}\,d\tau \tag{24}$$

In writing (22) in the form (23) it is implicitly assumed that

$$Div\, j = 0 \tag{25}$$

Now as a result of experiments indicating that moving charges in the statical sense produce effects similar to what we have regarded as electric currents, we conceive the likelihood of its being possible to express the currents in some manner which associates them with moving charges. What is next done is to replace the current j in expressions (23) and (24) by $1/c$ times $(\rho v+\dot{E})$, where c is at present a constant to be subsequently determined by experiment, and try whether an expression of this form for T, and a suitably chosen V can be made to serve as kinetic and potential energies which, by the application of Hamilton's principle, will lead to equations which represent experimental facts. The addition of the term $\dot{E}$, is more or less arbitrary, and is determined by the fact that anything which has to take the place of j and retain the feature which is characteristic of that quantity in continuous current theory, the feature of having zero divergence, must itself have zero divergence. To restrict ρv to have zero divergence would be to place a greater restriction on our analysis than we believe would be warranted by nature. For if v is the velocity of the charge density in the sense that, in a particle of finite size, the value of v at a plane multiplied by the density there represents the rate of disappearance of charge from one side, and if we are to retain the idea of indestructibility of charge, we must have the equation of continuity

$$-\frac{\partial \rho}{\partial t} = Div\, \rho v \tag{26}$$

so that, with the definition $\rho = Div\, E$, it is the divergence of $(\rho v+\dot{E})$ which is to be zero, and $(\rho v+\dot{E})$, or something proportional thereto, consequently suggests itself as the quantity to take the place of j in this analysis, although of course, as regards the mere requirement of zero divergence, we might add any other vector whose divergence is zero. The expression which is chosen for the potential energy is

$$V = \frac{1}{2}\int\int\int (E_x^2+E_y^2+E_z^2)\,d\tau \tag{27}$$

which is the same expression as is deduced for the case of electrostatics. The best that can be said for this, in the way of philosophical justification, is the rather weak statement that the potential energy ought not to depend upon how the field is produced. Thus, with T defined as

$$T = \frac{1}{2c}\int\int\int \{J_x(\rho v_x+\dot{E}_x)+J_y(\rho v_y+\dot{E}_y)+J_z(\rho v_z+\dot{E}_z)\}\,d\tau \tag{28}$$

where

$$4\pi J = \frac{1}{c}\int\int\int\frac{(\rho v+\dot{E})}{r}\,d\tau \tag{29}$$

and with V defined as above, and $Div\ J=0$, the Hamiltonian principle is applied with E and the displacement of the charge as coordinates connected by the restricting relation

$$Div\ E=\rho$$

with the result that equations (18) and (21) are obtained with H defined as

$$H=Curl\ J$$

which, since $Div\ J$ is zero is the equivalent of equation (17), thus showing this equation to be the equivalent of a definition of H.

The truth of whatever experimental facts are implied in equations (17) to (21) is of course the sole justification for the choice of the foregoing expressions for T and V.

Concerning other dynamical developments, reference may be made to the appendix to this paper, Note 2.

The significance of the equations and of the quantities occuring in them.—In reviewing a set of relations such as those under discussion,the question of which quantities are defined independently of the equations in which they occur and which are merely defined by those equations is one of considerable pertinence; and, the answer to the question may, even in a given set of relations, be different according to the point of view which is adopted as to the interpretation of relations. A definition of a quantity should be a statement of the sense in which it is to be used; and, it is desirable, if possible, that it contain a specification of the manner in which the quantity should be measured. In any case, even though infinite experimental skill be called for in the realization of the measure of a quantity in the strict sense of its definition, that definition should not call for the realization of conditions which are logically inconsistent with the laws of the subject which are being discussed. Thus, to define the magnetic field at a point inside an electron in terms of the couple on a magnet placed at the point is not merely to define it in a manner incapable of experimental realization on account of difficulty, but in a way which is logically meaningless. Moreover, in so far as the use of the equations, in their relation to the correlation of what is actually observed in experiments, never requires that we attempt to put a magnet inside an electron for the purpose of obtaining the data which are the ultimate goal of the investigation, it should certainly be possible to formulate what the equations have to say without introducing this meaningless concept.

In the process of tracing the analogy of our equations with those of some dynamical system, definitions do not raise such difficult questions, at least in respect of the dynamical arguments themselves. The purpose of the dynamical argument is to build a model satisfying the equations, and in doing this, we may define things like E and H in any way we happen to find convenient. Thus, to take an illustration which is purposely chosen in an artificial manner in order to bring out the point, let us suppose all space filled with particles, and let us denote by E the displacement of a particle of type A from its initial position as determined in some frame of reference. Let there be in addition other particles of type B, whose number per c.c. is denoted by ρ. Let there be some sort of binding between the particles so that the relation $Div\ E=\rho$ is always satisfied. Let v be the velocity of the particles of the type B as measured macroscopically in an element of volume. Then, if we choose quantities T and V as in (27) and (28), and develop the analysis in the manner there sketched, we shall of course arrive at the relations (17)–(21), with H defined as $Curl\ J$. It is not implied that the model arrived at is in any sense a true picture of the processes going on in the aether. The use of the particles in our thoughts about the model in no sense implies that there are, in the aether, particles of which E represents the displacement, and much less is it implied that the particles of which we have spoken are those fundamental but hypothetical particles to which, according to Poincaré, all dynamical systems may be referred. In fact, as already enforced, the dynamical argument does not constitute a proof of the relations, nor does it pretend to a unique discovery of an actual mechanism. The ultimate criteria for the truth of the relations must rest upon the experimental verification of such facts as are implied in the relations themselves, and the definitions of the quantities occurring in the relations need not, and in general can not, follow the definitions which were convenient for the dynamical analysis. Thus, it would be perfectly meaningless to define E as the displacement which a particle of the aether would suffer if the aether were composed of particles. Such a definition would moreover be useless in the sense that it would not be of a kind which would permit realization to the extent of allowing one to test the validity of any law in which the so-defined quantity occurred.

Another aspect of the difference in point of view appropriate to one who considers the dynamical development and one who considers the equations on their own merits is illustrated by the relation $\rho=Div\ E$. In the dynamical development, this was the condition of constraint introduced between two types of coordinates representing respectively the displacements of the particles of the A and B types. It was part of the law that these two things should be bound by a restricting relation.

From the point of view of one who considers only the final equations, however, the situation is slightly otherwise. If the nature of his measurements are such that he has, in terms of them, a perfectly definite definition of E, and if, in the process of calculating ρ, he always does so by the relation $\rho = Div\ E$, this relation becomes the definition of ρ. On the other hand, if he has some independent way of measuring ρ, and E, such a relation becomes part of the law. Again, if the experimenter decides upon some usable definitions for ρ and v, it may be that the equations (17)–(20) form largely a definition of the remainder of the quantities involved. It becomes immediately a question of interest as to how many of the quantities must be independently defined in order that the equations shall contain the definitions of the remainder. We shall presently probe this matter further in an endeavour to ascertain the most logical starting point for our definitions in order that those definitions shall represent the actual sense in which we use the quantities in our calculations. For the moment, however, it will be of interest to trace the meaning of the equations in the light of one who takes E and H as his fundamental starting points.

In the first place, (19) must be regarded as the definition of ρ, which, without such definition, would be meaningless, particularly in the case of moving charges. Equation (20) is merely a statement of the restriction of our interest to those problems in which this relation holds, and is not contained in the other equations as they stand. Put physically it is the analytical statement that we believe that no magnetic poles occur in nature.

From the stand point of one who takes E and H as his fundamentals, it can be shown that equation (17) is largely a definition of v, which, in itself, and without specific definition, would have no real meaning inside an electron for example. For further extension of this idea see appendix, Note 3.

It is important to realize that equations 17-20 give the fields when the motions of the charges are given, but they do not say anything about how the charges move as a result of the fields in which they exist.

It is convenient to express the solutions of the equations in terms of potentials slightly different from those used in their dynamical derivation. The desired solutions are

$$H = Curl\ U \tag{30}$$

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} - Grad\ \varphi \tag{31}$$

where

$$4\pi\varphi = \iiint \frac{[\rho]}{r}\,d\tau \quad , \quad 4\pi c U = \iiint \frac{[\rho v]}{r}\,d\tau \tag{32}$$

the possible values of the v's to be associated with the ρ's being restricted by the condition

$$-\frac{1}{c}\frac{\partial\varphi}{\partial t}=Div\ U. \qquad (33)$$

The square brackets are to be taken as indicating that the potentials are to be calculated in the usual retarded sense. They are the symbols of what may be interpreted physically as a finite rate of propogation of effects.

The vital importance of the term $\dot{E}$ in (17) is illustrated by the fact that, if it be omitted, expressions for E and H are obtained exactly like those given by equations (30) to (33), but with the retarded feature as symbolized in the square brackets absent. In other words, the physical language appropriate to that case is the language of instantaneous propagation.

Happily the equations representing φ and U and consequently E and H are made up of individual contributions from the volume elements in space. It is therefore possible to attach a meaning to the idea of the field due to one piece of electricity as distinct from that due to another. This is of course purely a matter of definition in the sense, for example, that if we have two charges A and B in each other's vicinity, and wish to speak of the field at a point P due to A alone, we decide to mean by this the portions of the total E and H which are contributed by those portions of φ and U which are contributed by the charge A alone.

Attention must be called to one further point in these solutions. Equations (30) to (32) enable us to calculate the fields provided that we assign the motions of the charges, and we could calculate from them fields whatever motions we assigned. Equation (17), however, contains the equation of continuity, and forbids our assigning motions which would in themselves violate this condition. They forbid, for example, that we set a problem in which v is everywhere zero, and ρ gradually diminishes to zero from a finite value. Equation (33) is the analytical guardian of the equation of continuity in forbidding us to assign in (32), values of ρ and ρv which would violate it. The equation of continuity does not of course follow as an absolute logical necessity without need of analytical specification. It does not apply to everything. It does not apply to the people in this country. The increase of population per year is not equal to the difference between the number who come into the country and the number who go out in that time, for some are born here, and some die here. Electricity might die, and it requires analytical specification to keep it alive.

We have remarked that equations 17–20, or their solutions, give the fields when the motion of the charges are given, but that they do not give the motions of the charges themselves in terms of the fields. This is the duty of the force equation. It would seem that, in accordance with the manner in which this law comes out of the dynamical analysis, it states that the electricity must move in such a way that the total force as measured by $E + [vH]/c$ must be zero for each point. In the form in which the result is used, however, in its application to an electron, it takes the less restricted form that the electron moves in such a way that the value of ρ times this vector integrates to zero when evaluated for the whole of it. Calling this vector a force, which name in itself does not of course imply anything about it to justify surprise in case the language used in connection with the said force should sound strange if considered in relation to ordinary mechanics, our law assumes the form that the electron moves in such a way that the total force which it exerts on itself due to its motion is equal and opposite to the force on it due to the external field. In other words, the law assumes the form

$$\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau+\iiint\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau=0 \qquad (34)$$

where E_1 and H_1 refer to the field produced by the electron itself at points inside it, and E_2 and H_2 refer to the field due to external causes. Before the force which the electron exerts upon itself, as a result of its motion, can be calculated, it is necessary to make a hypothesis as to what an electron is, as to how ρ is distributed within it, and as to how its shape and charge distribution change with the velocity. In the theory itself there is nothing to indicate this. Before pursuing this matter any further in a manner implying the existence of electrons, it will be well to consider the status which the equations attain immediately we specify any law of motion, i.e., any law stating a relation between E and H at a point, and the velocity, acceleration, rate of change of acceleration etc. at that point.

Foundation of electrodynamic laws in relation to actual observation.—Suppose in fact we write down some such equation as

$$\psi(\rho, v, \dot{v}, \ddot{v}, \text{etc.}) = E+\frac{[vH]}{c} \qquad (35)$$

where the function ψ is supposed to be specified, and where we have not written the right hand side in the form shown as a result of any general necessity for this form involved in what we wish to say, but simply in order to use in the illustration something which is not unnecessarily artificial and far removed from the facts as they turn out in practice. Directly we have some such equation as (35), we are in a position to formulate in one statement all that electromagnetic theory needs as verification from experiment in order that everything else may follow as a mere matter of definition.

Remembering that nobody can see a charge or measure its velocity with a meter stick, it will be well to avoid limiting the statement of the experimental law by the artificial conception that these things can be measured. We shall consequently formulate the law in such a way as to avoid these objections. We must suppose that we have decided upon some equation such as (35) which we are going to assert that experiment will justify. Let us write this equation down, replacing *E* and *H*, however by the values given by (30)–(32) so that the equation assumes the form

$$\psi(\rho, v, \dot{v},\ \text{etc}) = -\frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int\!\!\int\!\!\int\frac{[\rho v]}{r}\,d\tau$$
$$-Grad\int\!\!\int\!\!\int\frac{[\rho]}{4\pi r}\,d\tau + \left[\frac{v}{4\pi c^2}\,Curl\int\!\!\int\!\!\int\frac{[\rho v]}{r}\,d\tau\right] \tag{36}$$

it being understood that *c* is a constant number. The large square bracket indicates a vector product, while the small square brackets indicate retarded potentials. Let us also write down the equation of continuity

$$-\frac{\partial\rho}{\partial t} = Div\ \rho v \tag{37}$$

We may then express the general law as follows:

> It is always possible to find ρ and v as functions of position and time, and solutions of (36) and (37), such that the motion of the boundary of any body, beam of luminosity, etc. of macroscopic size (and which represents what is ultimately observed in the experiment), may be regarded as given by the value of v as obtained for the points of the boundary, this value being, however, smoothed out in the macroscopic sense. In the case of a statical boundary, the position is to be determined either from the time integral of the velocities, or by the criterion that ρ suffers a discontinuity there in the macroscopic sense. F

It must be pointed out that all observable phenomena are included as particular cases of this law. Thus, if a beam of electrons enters a Faraday cylinder, and charges an electrometer, the needle ultimately moves; and, in the full generality of the case, this motion may be described in the above manner. Again, even though the observer should base his conclusions upon measurements of the path of a beam of light produced by the passage of a beam of electrons through an ionized gas, the boundary of this beam is really something whose position may be regarded as specified in the above sense. Again, if one is discussing the amount of scattering of a beam of electrons in passing through a sheet of matter, the movement of the needle of the electrometer is ultimately to be regarded as determined by v at its various parts, this v being bound up with the v's and ρ's in the electrons of the atoms which are causing the scattering, in such a way that the whole set of motions and densities satisfies equations (36) and (37).

Redefinition of the field vectors.—Equation (36) does not contain E or H, but being of a rather cumbersome form, it may be a convenience to introduce certain quantities φ, U, E, H, *defined* as follows:

$$4\pi\varphi = \iiint \frac{[\rho]}{r} d\tau \quad , \quad 4\pi c U = \iiint \frac{[\rho v]}{r} d\tau \tag{38}$$

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} - Grad\ \varphi \tag{39}$$

$$H = Curl\ U \tag{40}$$

Then, in view of the restriction which was specified in (37) and which has controlled the relation between ρ and ρv to some extent, the above definitions alone insure the following relations

$$\frac{1}{c}\left(\rho v + \frac{\partial E}{\partial t}\right) = Curl\ H \tag{41}$$

$$-\frac{1}{c}\frac{\partial H}{\partial t} = Curl\ E \tag{42}$$

$$Div\ E = \rho \tag{43}$$

$$Div\ H = 0 \tag{44}$$

It is important to realize, however that E and H have now been relegated to the role of subsidiary quantities introduced for convenience of analysis, but involving nothing about unit pole or unit charge, and the equations connecting them result from their *definitions*, the sole law to be sought from experiment being contained in the statement F made above.

Having reduced E and H to their proper degree of subordination it is not the purpose to discard them, or even to deny ourselves the physical satisfaction of visualizing them in terms of an aether. Whenever they become too assertive, however, by demanding something on their own account, we shall know how to assure them of their proper position in relation to the true representatives of the law, equations (36) and (37).

The value of the field vectors.—In spite of the apparent artificiality of the law as given above, a little consideration will show that a law of this form is what we actually use in all of our calculations. We make use of the properties of substances for the purpose of making short cuts in our calculations, and restricting the generality of our solutions, but would readily be willing to admit that these are only used for convenience and do not figure as parts of the general law. Thus, in the problem of the electrometer and Faraday cylinder cited above, we find it of enormous convenience to recognize that the Faraday cylinder, quadrants of the electrometer, and connecting wires are what we call conductors. But what is a conductor? It is something in which the quantity E defined above is zero for stationary conditions. This criterion enables us very quickly to determine the distribution of ρ over the Faraday cylinder, electrometer quadrants etc., and to be assured of the uniqueness of our solution on the basis of the ordinary uniqueness theorem for electrostatics. Our restriction to electrostatics as regards the Faraday cylinder and electrometer, our belief that those oscillations which were started when the electricity rushed into the quadrants will not persist for a long time, is again something which the general law cannot be expected to tell us, so again we invoke a special property of materials viz. resistance, which insures that, provided we wait long enough, we shall really arrive at a statical distribution in the electrometer. Then, as regards the motion of the needle, in the full generality of the case, this needle would be regarded as distribution of electronic density of a complicated kind. Its very moment of inertia would be sought in the electromagnetic masses of its molecules. The torsion in its fiber would find its representative in electromagnetic phenomena. In fact, the resultant motion of the needle as a whole could really be described in the sense implied in our general law without the invocation of moments of inertia, fiber torsion, etc. Here, however, we make use of our practical knowledge of matters not pertaining to the general law to simplify the procedure. The study of mass conceived as having an electromagnetic origin enables us to surmise that provided the needle does not move so rapidly that the velocity of its parts become comparable with that of light, its angular acceleration is proportional to the couple on it as determined from the electrical forces in the sense ordinarily understood[1], in combination with the torsion of the fiber as measured mechanically. It is in order to specify these characteristic restrictions which are peculiar to matter, but are not part of any general law, that the introduction of the quantities E and H finds one of its greatest fields of usefulness, for it is in terms of them that these restrictions become expressed. While it is convenient and permissible to use these subsidiary properties of conductivity, etc., it must be clearly recognized that their use is only justified in so far as they are subservient to the general law of the type (36)-(37). The ultimate solution may satisfy certain special conditions arising from the existence of conductivity, etc., in the parts of the apparatus, but as far as a general law of nature is concerned the essential thing is that it shall conform to some such conditions as (36)-(37).

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[1] It must be pointed out that for the case where the equation of the type (35) is a special case of (34), and in view of the definitions of E and H given in (38)–(42), there is an ambiguity as regards the value of ρ at a point in any given problem. In fact, an increase of ρ in the same proportion throughout will not alter matters as concerns the solution sought. If, however, in examining the motion of any of the material bodies which happens to move with small velocity, we calculate the mechanical forces on it from the accelerations, we have a means of fixing the scale of magnitude of ρ by making the electrical force equal to the mechanical. It is of course only for the purpose of correlating the electrical forces with mechanical forces as ordinarily defined that it becomes necessary to fix the scale of ρ at all. Once the scale of ρ is fixed it is but a short step to the realization of the ordinarily defined unit of charge. When this is done, it is only a small step to show that c represents the ratio of the electromagnetic unit of current as defined in terms of the forces between currents to the electrostatic unit as defined in the manner cited above. As a matter of fact it is not even necessary to definitely fix the meaning of the electrostatic unit before showing that c represents the ratio of the units, whatever be the choice for the absolute magnitude of the electrostatic unit.

In case any one should demur against the complete relegation of E and H to the role of subsidiary quantities defined by equations (38)–(40), we may point out that it is only in the sense of such a definition that we actually *use* them in many of the problems which are of most importance to us. Thus, suppose we should wish to determine by experiment how the field E varies with position within an electron of the atom of a certain metal, how should we do it? We should, certainly, not attempt to put a unit charge there. We should make up a hypothesis as to the distributions of the electrons and of their velocities and of the values of ρ and v within the electrons. We should write down the relations of the type (36) and (37) which we accepted as the true ones, using possibly the subsidiary scaffolding (17) to (21) for mathematical convenience, and we should try, by means of the distribution of ρ and v which we had assumed, to predict some macroscopically observable phenomena in terms of which we could make our test. We should adjust our theory of the ρ's and v's until we did secure a system which represented all the facts which we could observe, and having found this theory, we should proceed to calculate our fields inside the electron in terms of the definitions given by equations (38)–(40). It is true that our solutions for ρ and v might not be unique as regards their power to represent the observable phenomena; but, they would be unique in the practical sense that they agreed with our observations as far as we were able to make the test.

It may be urged that we may determine the field inside an atom, for example, by shooting an electron through it and calculating the deviation in its path. The difficulty arising here is concerned with the fact that the field which is defined in terms of the motion of the electron as a whole, is too coarse-grained a quantity to serve as applicable to a point inside the electron. Nevertheless, there are certain interesting aspects of the definition of a field in terms of the motion of an electron as a whole, for which reference may be made to Appendix, Note 4.

The degree of fundamentality of the circuital relations.—In line with the above standpoint as to the subordinate positions of E and H, and of the circuital relations themselves, since these are determined by the forms of the definitions of these quantities, we may have a suspicion as to how far these relations express what is most fundamental in the whole subject. Such a suspicion is entirely justified, and to illustrate the extent to which it is justified, we shall show that it is possible to make relations of the type of the circuital relations play a role in the description of a highly arbitrary law.

Let us cease to think of all electrical phenomena, and divide all space into a net-work of infinitesimal boxes which may, in the initial instance, be rectangular if we wish. Let us now write a set of perfectly arbitrary numbers on cards, and throw one into each box. Let the boxes now move about in any manner we like to assign. Let them twist into all sorts of shapes, some becoming very much elongated and others flattened, subject only to the condition that the network of boxes always remains unbroken, and that the boxes were originally chosen so small that all their dimensions remain infinitesimal. It will also be convenient to have the dimensions of these boxes of an order of magnitude lower than that of the differential elements of volume which are concerned in the specification of the ultimate differential equations of the medium. It will be obvious that while there may be some law according to which the sum of the numbers assigned to any small volume of space may be expressible in terms of the distributions of the numbers throughout space and of the motions of the boxes, the law will be entirely of our own creation. Nevertheless, if we let W represent the volume of some box which contains a number N, and if we define a quantity ρ by the equation

$$\rho=\frac{\Sigma N}{\Sigma W}$$

where ΣW is taken for an element of volume $d\tau$, and ΣN represents the corresponding summation of the numbers therein, and if we define two quantities φ and U by the equations

$$4\pi cU=\int\int\int\frac{[\rho v]}{r}\,d\tau,\quad 4\pi\varphi=\int\int\int\frac{[\rho]}{r}\,d\tau$$

where v is the velocity of a box, and c is a constant; and, if we then define two quantities E and H by the equations

$$E=-\frac{1}{c}\frac{\partial U}{\partial t}-Grad\ \varphi,\quad H=Curl\ U$$

the quantities E and H so defined will satisfy the circuital relations, for the necessary relation (33) which must exist between U and φ for this to be true, and which is the analytical equivalent of the equation of continuity, is automatically taken care of by our having specified that the numbers in the boxes are to remain unchanged as the boxes move about. Having done all this it *might* of course be possible to express the law of the motion of the numbers at any point in terms of the fields E and H, in a manner analogous to that adopted in the force equation of electromagnetic theory, although the equation would naturally be quite different from that equation in form. It is rather disturbing to contemplate the fact that any other being who happened to discover this purely arbitrary law of our own creation might discover these circuital relations as playing an important part in its description, and might be led to regard the fact as very fundamental. It would seem that we have here an illustration of the fact that what we call the laws of nature are in very large measure the statement of the way in which we have decided to think about nature. Indeed, the whole claim of E and H to fundamentality in electro-magnetic theory lies in the fact that it is possible to describe the motion of an element of charge entirely in terms of itself and of E and H, in a fairly simple manner, without any specification of the motions of the remaining electrons, other than is involved in the specifications of E and H themselves.

Without pursuing in too great detail this rather futuristic way of thinking, we are driven to admit that the equations giving the fields in terms of the motions of the charges contain practically nothing which is not definition, and that the real law of electro-magnetic theory is not stated until the force equation is stated. So far we have spoken of this equation without any references to the hypothesis of electrons, except in so far as we have spoken of electrons for purposes of description.

The force equation in relation to electrons.—As already implied, the force equation is usually stated in the sense implied in the statement that the force exerted on the electron due to its motion is equal and opposite to the force due to the external field. Directly some hypothesis has been made as to the variation of shape, density, etc. with velocity, as for example the hypothesis which has been made in the electron suggested by Prof. Lorentz, it is ideally possible to calculate the force which the electron exerts upon itself. It turns out to be zero for all cases of uniform velocity, but it contains terms involving the acceleration of some specified point, (the center for example in the case of an electron of symmetry) and all the higher time derivatives of the motion. Since there is no reason to expect that, in the general case, the resultant acceleration of the electron will be in the direction of what we have artificially called the force which it exerts upon itself, our law of motion will, in general, when expressed vectorially in terms of the accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ in directions parallel to and perpendicular to the velocity assume the form

$$\text{External force} = m_1\,\mathbf{a}_1 + m_2\,\mathbf{a}_2 + \text{terms involving higher time derivatives etc.} \qquad (45)$$

the quantities m_1 and m_2 which are in general functions of the velocities are called the longitudinal and transverse masses, so that by this analytical artifice, what was originally a very unusual way of expressing an equation of motion of an electron has been forced into something like the familiar form

$$\text{External force} = \text{mass} \times \text{acceleration}. \tag{46}$$

The experimental verification of any hypothesis with regard to the electron rests upon the determination of the force which it exerts upon itself due to its motion, and is usually restricted to the determination of the masses in equation (45), and usually the transverse mass at that. It would appear that here our experimental knowledge of the situation is most incomplete. The postulation of the restricted theory of relativity, on the bases of experiments supporting it, carries with it the conclusion that the longitudinal and transverse masses m_1 and m_2 are of the form

$$m_1 = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \tag{47}$$

$$m_2 = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \tag{48}$$

the second of which expressions has been experimentally verified by the experiments of Bucherer. These are of course the masses of the Lorentzian electron; and, for the case of axes which coincide with neither the tangent nor the normal to the path, the equation of motion is

$$E_2 + \frac{[uH_2]}{c} = m_0 \frac{d}{dt}\left(\frac{u}{\left\{1-\frac{(u_x^2+u_y^2+u_z^2)}{c^2}\right\}^{\frac{1}{2}}}\right) \tag{49}$$

to the extent that the higher derivatives referred to in equation (45) are negligible. Equation (49) is of course consistent with the values given by (47) and (48) for the longitudinal and transverse masses. It is to be observed that while the postulation of the restricted theory of relativity provides us with some information as to the equation of motion of the electron, it does not give this equation completely. If we should assign some form for the force which the electron exerts upon itself in terms of its acceleration, rate of change of acceleration, and all higher derivatives, when the velocity is zero, or if we should do the equivalent of this by assigning the nature of the electron and the mutual relations of the higher time derivatives of the motion of its various parts, supposing this relation to be unique, then, the theory of relativity would have the power to tell us what the said force exerted by the electron upon itself would be when the velocity was not zero. Apart from this, however, it would be powerless to suggest anything as regards the part played by the higher time derivatives. (See Appendix, Note 5.)

The equation of motion of the electron remains the most important thing requiring experimental investigation, and the part most in need of it. It is the key to the whole situation, and is yet something of which we know least. Of course for slow motions, and for fields uniform over the electron, there is no difficulty, but, for high speeds, or more particularly rapid changes of speed, and sharply varying fields it would seem that we lack much. In particular, attention may be called to one case.

It is possible to obtain in the laboratory electronic speeds so high that the field of the electron tends to become concentrated in an equatorial plane. Suppose the velocity is so great in some particular case that the variation of the field is appreciable over distances from the equatorial plane comparable with the diameter of an electron. One's whole physical instinct revolts against saying that the electron upon which this field acts will, for any given state of motion of its center, adjust its shape in exactly the same way as if the external field were uniform over it. The process of determining the motion by equating the integrated external force to the integrated internal force seems now extremely crude. Yet, we have absolutely no theoretical criterion to guide us in assigning relative motion to parts of the electron, unless something could be developed on the lines of a postulate that the resultant vector $(E+[vH]/c)$ must be zero at each point. To attempt to apply ordinary mechanical concepts to the inside of the electron is rather like trying to explain the tenacity of a brick by considering it as made up of a number of houses.

THE EQUATIONS OF CONSERVATION OF MOMENTUM AND ENERGY

Significance of the equations.—In view of the very fundamental role played by momentum and energy in the discussion of some of the most vital aspects of modern physics, a word concerning these quantities will not be out of place. In the first place it must be emphasized that the equations representing the principles of conservation of momentum and energy are not founded upon any experimental basis as to the fundamental necessity for the conservation of those quantities. Given the electromagnetic equations they follow as a result of pure algebra, and their derivation is really the equivalent of finding the quantities which must be called work, energy-density, and momentum, in order that the quantities so defined shall be related by the same equations as those which relate the correspondingly named quantities in ordinary dynamics. The only justification for naming the quantities $[E.H]/c$, $(E^2+H^2)/2$, $\rho(v.\,\overline{E+[vH]}/c)$, density of momentum, density of energy, rate of doing work per unit volume, is to be found in the fact that the electromagnetic equations give to these quantities the properties which the correspondingly named quantities in ordinary dynamics are supposed to possess. It may happen, and actually does happen in the case of work and energy, that the quantities to which these names are assigned are chosen as the result of the consideration of a rather specialized type of problem. Having once named them one is inclined to seize upon them as quantities of fundamental importance. The glorious and honorable titles work, energy and momentum which have been assigned to them tend to cause us to forget their birth, and to endow them, on the basis of these titles, with properties which they never obtained from their parents. Then, when we consider some problem which is of a more general type than that which was in mind when the names were given, and find that these quantities do not have the properties which we expected of them, we are apt to be greatly astonished and claim that our theories are doing something outrageous in calling for a revision of the principle of the conservation of energy. The truth of the matter of course lies in the fact that if in any special case we should find any discrepancy of the kind stated, it simply means that, to the degree of generality implied by the inclusion of that special case, the quantities in question had no right to have the names work, energy, etc., assigned to them. The only proper basis for choosing the quantities which shall be given these names is that they shall be the quantities to which our scheme of laws gives, in all cases, the properties which we desire of them.

Localization of energy.—Difficulties are also apt to arise from too strong visualization of the supposed seat of the energy in the medium. Thus, to take the very simplest case, that of electrostatics, we recall that it is customary to say that the energy is distributed throughout the volume to the extent $E^2/2$ per unit volume. The origin of this conclusion may be summarized as follows: We first consider a system of charged bodies, and assuming only that the forces are derivable from a potential V we calculate for the energy the expression $W=\frac{1}{2}\Sigma QV$. Extending this to the case of continuous distributions with volume densities ρ and surface densities σ it becomes replaced by

$$W=\tfrac{1}{2}\int\int\int \rho V d\tau+\Sigma\tfrac{1}{2}\int\int \sigma V dS \qquad (50)$$

The next thing is to replace ρ by Div E, which, combined with the assumption that E has no curl, is the equivalent of assuming the law of inverse squares. On applying Green's theorem to the volume integral thus formed, a new volume integral $\frac{1}{2}\int\int\int E^2 d\tau$ is formed, and a surface integral is thrown out at the boundary of each of the regions which are bounded by the surfaces. With the exception of the bounding surface at infinity, each surface gives rise to two surface integrals, one on account of that surface functioning as the outer boundary of the region which it incloses, and the other on account of its functioning as the inner boundary of the region which lies outside it. These two surface integrals when combined so as to give one surface integral for each surface turn out to just cancel individually the several surface integrals of equation (50). They do this on account of the fact that σ is just equal to the discontinuity in E_n at a point on the surface. The result of all this is that the volume integral $\frac{1}{2}\int\int\int E^2 d\tau$ remains as the sole representative of the energy. The form of this expression strongly suggests the assignement of energy $E^2/2$ to each unit of volume, and so strong is this suggestion that undoubtedly a good many physicists regard the demonstration as *proving* that the energy is to be so located, or at least regard any objection to the acceptance of the demonstration as a mere splitting of hairs. The expression representing the energy entirely as a volume integral throughout space is not confined to the case where there is a finite volume density, but holds even in the case where the charges are entirely confined to the surfaces. In this case, of course, the volume integral of (50) is zero, but it can nevertheless be split by Green's theorem into two parts, a volume integral $\frac{1}{2}\int\int\int E^2 d\tau$, and a set of surface integrals which cancel the surface integrals of (50) leaving the expression $W=\frac{1}{2}\int\int\int E^2 d\tau$ as before.

In order to demonstrate the artificiality of the process by which the above conclusions as to the location of the energy are reached, it may not be out of place to consider an actual possibility as regards another method of distributing the energy.

Imagine a mechanism by which two bodies are connected by a sort of screw, and that, as each body moves along the screw towards the other, it winds up, by means of a nut engaging with the screw, a spring which is located within itself. A means is supposed to be provided for preventing the body itself from rotating. One body will of course experience a force as it is pushed towards the other, and it would not be impossible to arrange the pitch of the screw and the nature of the springs so that this force was that of the inverse square. We can further imagine a mechanism of this kind extended to each pair of bodies in our whole system. The actual mechanism would be, of course, extremely complicated in practice, but it would not appear that its realization would present any *logical* difficulties. Now if all this mechanism of screws and springs be rendered invisible, leaving only the bodies visible, and if the whole system be presented to somebody else who knows nothing of the mechanism, he will proceed to study it in the same sort of way that we study electrical phenomena. He will find that the bodies exert upon each other forces which are derivable from a potential. This is all that he will need to construct the expression $W=\Sigma\frac{1}{2}QV$ for the energy. He will then extend this to $\frac{1}{2}\int\int\int\rho V d\tau$ to include the case of continuous distributions. He will discover that the forces are governed by the inverse square law, which is all that he needs to enable him to write ρ as proportional to Div E. He will then be able to carry out the transformation by Green's theorem arriving at the expression $W=\frac{1}{2}\int\int\int E^2 d\tau$, and will have proved possibly to his own entire satisfaction that the potential energy is located in the medium to the extent $E^2/2$ per unit volume whereas we, who have constructed the mechanism, know that in the sense that we understand the term location of energy, the energy is really in the springs which are hidden in the bodies and is not to be found in the medium outside of the bodies.

With these preliminary general considerations we pass on to consider a few specific points in relation to the equations of energy and momentum.

The momentum equation.—The momentum equation is expressed in the form

$$\int\int\int\rho\left(E+\frac{[vH]}{c}\right)d\tau = -\frac{\partial}{\partial t}\int\int\int\frac{[EH]}{c}\,d\tau + \text{Surface integrals}$$

the volume integrals being taken throughout any specific volume over whose surface the surface integrals are taken. $[E\,H]/c$ is the momentum per unit volume. The equation as it stands, however, is of very little use. Its strength lies in the fact that it applies to any field satisfying the electromagnetic equations. If then we incorporate a hypothesis such as that embodied in the statement that the electron moves so that the force exerted on it by the external field is equal and opposite to the force which it exerts on itself, and if we remember that both E_1, H_1, the field of the electron, and E_2, H_2, the external field are separately solutions of the electromagnetic equations, we shall be able to write

$$\int\int\int\left(E_2+\frac{[vH_2]}{c}\right)\rho\,d\tau = -\int\int\int\left(E_1+\frac{[vH_1]}{c}\right)\rho\,d\tau \qquad (51)$$

and,

$$\int\int\int\left(E_2+\frac{[vH_2]}{c}\right)\rho\,d\tau = \frac{\partial}{\partial t}\int\int\int\frac{[E_1H_1]}{c}d\tau + \begin{matrix}\text{Surface integral involv-}\\ \text{ing } E_1 \text{ and } H_1\end{matrix} \qquad (52)$$

In this form the equation is of use, since, when the type of the electron and its motion are given, the right hand side can be calculated as a function of the motion, and the equation therefore relates the motion of the electron to the external field. In this form the equation does of course involve something drawn from experiment, or at any rate from additional hypothesis, viz: the statement of equality of external and internal force. It is indeed the analytical equivalent of that law. One useful service performed by the equation in this form is that of enabling us to calculate the electromagnetic masses, which are defined as the coefficients of the accelerations in the directions parallel to and perpendicular to the path, in the expression for the external force acting upon the electron. The particular use of the momentum equation for this purpose over that involving a direct calculation of the force exerted by the electron on itself lies in the fact that while, in both cases, we strictly arrive at a result in which the external force is not completely expressible in terms of the velocity alone, all higher time derivatives being in fact present, the momentum equation by making use in the most direct manner of the assumption of the quasi stationary principle is able to arrive more directly at a result which is correct as far as the acceleration terms are concerned.

The quasi stationary principle amounts to stating that, in calculating the momentum

$$M=\int\int\int\frac{[EH]}{c}\,d\tau$$

we may, to the approximation cited, calculate it at each instant on the assumption that the field of the electron is the same as it would have been if the electron had been moving for an infinite time with the velocity which it has at the instant. On this assumption M is a function of v only

$$M=vf(v)$$

Hence, for the component parallel to the x axis

$$M_x=v_xf(v)$$

As regards the integral through all space we may replace partial $\partial/\partial t$ by d/dt , so that

$$\frac{\partial M_x}{\partial t}=\frac{dM_x}{dt}=\dot{v}_xf(v)+v_x\,\dot{v}f'(v)$$

If we choose the axis of x perpendicular to v, so that $v_x=0$, we have

$$\frac{dM_x}{dt}=\dot{v}_xf(v)=\left(\frac{M}{v}\right)\dot{v}_x$$

Hence the transverse mass m_2 is given by

$$m_2=\frac{M}{v}$$

If we choose the axis of x parallel to v, so that $v_x=v$

$$\frac{dM_x}{dt}=\dot{v}f(v)+v\,\dot{v}\,f'(v)$$

$$=\dot{v}\frac{d}{dv}\left\{vf(v)\right\}=\dot{v}\frac{dM}{dv}$$

showing the longitudinal mass m_1 to be given by

$$m_1=\frac{dM}{dv}$$

It is to be particularly remarked that it is only because the equations of motion of the electron have been forced into the form force = mass × acceleration that they have assumed this form at all; and, as already pointed out, we have no right whatever to expect any direct or simple relation between these particles which constitute the electrons and those hypothetical particles of constant mass to which, as Poincaré has shown us, any dynamical system may be referred.

The energy equation.—Remarks exactly analogous to the above may be made in relation to the energy equation[1]

$$\iiint\left(v.\,\overline{E+\frac{[vH]}{c}}\right)\rho d\tau=-\frac{\partial}{\partial t}\iiint\left(\frac{E^2+H^2}{2}\right)d\tau+c\iint[EH]_n dS \quad (53)$$

Here again, the equation applies to the field of the electron alone as well as to the whole field, so that if subscript unity refers to the field of the electron we have

$$\iiint\left(v.\overline{E_1+\frac{[vH_1]}{c}}\right)\rho d\tau=-\frac{\partial}{\partial t}\iiint\left(\frac{E_1^2+H_1^2}{2}\right)d\tau$$
$$+c\iint[E_1H_1]_n dS \quad (54)$$

If, *and only if*, all parts of the electron move with the same velocity we are at liberty to take the v outside of the integral on the left hand side, and then, by utilizing the fundamental assumption as to the equality of the external force on the electron and the force which it exerts on itself, we arrive at the relation

$$v\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau=\frac{\partial}{\partial t}\iiint\left(\frac{E_1^2+H_1^2}{2}\right)d\tau-c\iint[E_1H_1]_n dS \quad (55)$$

which expresses a relation between what we may call the work done by the external forces on the electron, the increase of energy of the field of the electron, and the flow of energy of the field of the electron through the boundary.

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[1] As pointed out in Note 2, of the appendix, and also earlier in this report, the expression $\frac{1}{2}\int\int\int(E^2+H^2)\,d\tau$ is not the only one which is used in electrodynamics as the appropriate expression for the energy.

It is to be observed, however, that the whole justification for calling the left hand side the work done per second on the electron by the external forces, and for the expectation of any relation of the above kind between the work as so defined and the quantity which has been called the energy of the electron, rests upon the possibility of taking v outside of the integral of equation (54). It is only for such a case, i. e., one where the velocity of all parts of the electron is the same, that the names work and energy are appropriately given to the quantities to which they have been applied.

The Lorentzian electron and the conservation of energy.—Now it is usually argued that in the case of the Lorentzian electron, a difficulty exists as to the application of the principle of conservation of energy; and the difficulty is this:

If we calculate the longitudinal electromagnetic mass of the electron we find

$$m_1=\frac{e^2}{6\pi ac^2}\left(1-\frac{v^2{}_0}{c^2}\right)^{-3/2} \tag{56}$$

where a is the radius of the spherical shell of charge which constitutes the electron when at rest, v_0 the velocity of the center of symmetry of the electron, e the electronic charge, and c the velocity of light. The external force parallel to the velocity is obtained by multiplying the above value of m_1 by dv_0/dt, and the quantity which is called the rate of doing work by, or activity of, the external force and is obtained by multiplying the result by v_0 is

Activity of external force =

$$v_0\int\int\int\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau=\frac{e^2}{6\pi ac^2}\left(1-\frac{v_0{}^2}{c^2}\right)^{-3/2}v_0\frac{dv_0}{dt} \tag{57}$$

On the other hand the rate of increase of the energy of the electron as calculated from the first member of the right hand side of (54), with the boundary extended to infinity is

Rate of increase of energy of electron =

$$\frac{e^2}{6\pi ac^2}\left(1-\frac{v_0{}^2}{c^2}\right)^{-3/2}v_0\frac{dv_0}{dt}+\frac{e^2}{24\pi ac^2}\left(1-\frac{v_0{}^2}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt} \tag{58}$$

Expression (57) is less than (58)[1] by

$$\frac{e^2}{24\pi ac^2}\left(1-\frac{v^2{}_0}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt},$$

and the inference is that the Lorentzian electron violates the principle of conservation of energy. With the object of removing this difficulty, Lorentz has gone so far as to invoke the energy associated with certain non-electrical forces first introduced by Poincaré,[1] for the purpose of accounting for the equilibrium of the electron in its spherical and flattened forms.

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[1] The discrepancy has nothing to do with any possible value of the third integral in (54) when taken over the infinite boundary or with the application of the so called quasi stationary principle.

Now when we recall that the principles upon which (57) is obtained involve nothing but the momentum equation in the form (52), combined with the *definition* that the activity of the external force is to be taken as the external force multiplied by the velocity of the center of the electron, and when we also recall the origin of the expression for the energy, we see that an inconsistency is algebraically impossible provided that all the quantities are correctly named. The fact is that, in the case of the Lorentzian electron, we have a situation where all parts of the electron do not move with the same velocity. The electron contracts while in motion, so that, in the case of acceleration in the direction of the velocity, for example, its fore end moves at each instant slower than its center and its back end moves faster. The situation is one where we are not justified in taking v outside of the integral in equation (54) and we are consequently unable to make use of the equality of the external force and the force which the electron exerts upon itself to obtain an equation such as (55). As a matter of fact the product of external force and velocity of the center of the electron is a quantity which our equations do not endow with any very important property in the case of the Lorentzian electron, however much its form may tempt us to expect important properties in it; and it is important to realize that there is no other source from which this quantity may expect to derive important properties.

It will perhaps be of value to probe this matter in detail in order to ascertain just what the equations do have to say regarding it. Using as before E_1 and H_1 to denote the field of the electron, equation (54) follows as an algebraic necessity from the electromagnetic equations. Let us put

$$v = v_0 + \Delta v$$

where v_0 is the velocity of the center of symmetry of the electron, and Δv is the algebraic excess of the velocity of a point on the surface over the value v_0. We then have,

$$\int\int\int\left(v.\, \overline{E_1 + \frac{[vH_1]}{c}}\right)\rho d\tau = \int\int\int v_0\left(E_1 + \frac{[vH_1]}{c}\right)\rho d\tau + \int\int\int \Delta v\left(E_1 + \frac{[vH_1]}{c}\right)\rho d\tau$$

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[1] Sur la Dynamique de l'electron, Rendiconti del Circolo Mathematico di Palermo 21, p. 129, 1906.

It is now possible to take v_0 outside of the integral in which it occurs, and we may then legitimately make use of the equality of the external force on the electron and the force which the electron exerts upon itself, and may write

$$\iiint\left(v.\,E_1+\frac{[vH_1]}{c}\right)\rho d\tau=-v_0\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau+$$
$$\iiint\Delta v\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau$$

substituting in (54) we find

$$v_0\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau=\iiint\Delta v\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau$$
$$+\frac{\partial}{\partial t}\iiint\left(\frac{E_1^2+H_1^2}{2}\right)d\tau \qquad (59)$$

where we have omitted the surface integral in (54) on the basis that it is taken over an infinitely distant boundary.

Thus we see that the so called activity of the external force as represented by the left hand side of (59) should *not* be equated to the rate of increase of energy as represented by the second term on the right hand side of (59). Equation (59) is not a denial of the conservation of energy, it is simply a criterion as to what should be called work and energy in order that these quantities shall be related in the manner which their names signify, and the equation shows us that the member on the left hand side is not the quantity to call the rate of doing work on the electron by the external force if the second member on the right hand side is to be called the total rate of increase of energy of the electron.

If we calculate the magnitude of the first member on the right hand side of (59), we find for it (see Appendix, Note 6) the value

$$-\frac{e^2}{24\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt}$$

so that, in view of the value which has been quoted in equation (58) for the second member on the right hand side, we have

$$v_0\iiint\left(E_2+\frac{[vH_2]}{c}\right)\rho d\tau = -\frac{e^2}{24\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt}$$
$$+\frac{e^2}{6\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-3/2}v_0\frac{dv_0}{dt}+\frac{e^2}{24\pi ac^2}\left(1-\frac{v_0^2}{c^2}\right)^{-1/2}v_0\frac{dv_0}{dt}$$

Thus the term which has arisen from the member of (59) which involves Δv just cancels that portion of (58) which has been the subject of all the discussion, leaving an equation which is perfectly consistent with equation (57) which was built up from the momentum equation. Of course it is algebraically impossible that the result could have been otherwise.

MATERIAL MEDIA

The Lorentzian equations.—The impossibility of dealing with each individual molecule in a medium naturally makes it desirable to formulate equations which express the laws as far as possible in terms of quantities which have a significance only in the macroscopic sense. With this end in view, Lorentz[1], starting with the field equations for free aether has proceeded to build up average values for the quantities E, H, ρv, etc., occurring in the equations, and has arrived at the following system of equations applicable to a moving material medium:

$$\frac{1}{c}\left(\frac{\partial E}{\partial t}+\rho u+j+\frac{\partial P}{\partial t}+Curl\,[Pu]+c\,Curl\,m\right)=Curl\,B \tag{60}$$

$$-\frac{1}{c}\frac{\partial B}{\partial t}=Curl\,E \tag{61}$$

$$Div\,B=0 \tag{62}$$

where the quantities have the following significance:

- j is the conduction current density.
- u represents the velocity of the material medium at any point.
- ρ is the net density macroscopically considered in a macroscopic volume element.
- E and B are the average values of the true electric and magnetic fields, the averages being taken throughout the macroscopic element of volume.
- P is the polarization. In the Lorentzian sense, it is defined in a way which is rather different from what one would expect on the basis of an analogy with Poisson's theory of dielectric media at rest.

P is in fact defined in the following way: Let us divide the electricity in a macroscopic element of volume into its positive and negative parts which we shall keep separate in our mind's eye, making no attempt to associate any one positive element with some neighboring negative element for the purpose of forming a doublet. Multiply the macroscopic density ρ_0 of the positive electricity by the amount r by which it has been displaced from some standard position, and call $p_+=\rho_0 r$ the polarization corresponding to the positive electricity. Build up a similar quantity for the negative electricity. Then, vectorially

$$P=p_+ + p_- \tag{63}$$

The characteristic feature of this polarization which distinguishes it from that defined in terms of doublets, is that it speaks of the density of the positive and negative electricity at *one and the same point* (macroscopically considered), and concerns itself with the amounts by which they *have been* displaced *to* that point, whereas the point to which a doublet is referred is the point *from which* the charge has been displaced. This characteristic difference has a very important bearing upon the manner in which it is possible to average the ρv terms.

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[1] Enzyk. der Math. Wiss. v. 2, pp. 200–9.

The quantity m, the magnetization, is defined as

$$m=m_1+m_2 \qquad (64)$$

where

$$cm_1=\frac{1}{2}\rho_0\,[r_1\,\dot{r}_1] \text{ and } cm_2=\frac{1}{2}\rho_0\,[r_2\,\dot{r}_2] \qquad (65]$$

If we define an additional quantity D by the relation

$$D=E+P \qquad (66)$$

this quantity satisfies the relation

$$Div\ D=\rho \qquad (67)$$

It is also customary to define a quantity h, by the relation

$$h=B-m \qquad (68)$$

Now an examination of the definitions of the above quantities will show that, with the exception of perhaps u, not one of the quantities is capable of experimental realization in terms of its definition. In other words, we could not utilize the definitions for the measurement of the quantities. In view of this, it is a matter of importance to inquire just what the equations mean. We can get some light upon this matter by going through a process of solving them analogous to that adopted in the equations for free aether. If we do this, we can show without difficulty that we arrive at the relations

$$E=-\frac{1}{c}\frac{\partial U}{\partial t}-Grad\ \varphi \qquad (69)$$

$$B=Curl\ U \qquad (70)$$

where U and φ are of the same form as (32) except that in the expression for U, ρv is replaced by

$$J=j+\rho u+\frac{\partial P}{\partial t}+Curl[Pu]+c\ Curl\ m \qquad (71)$$

and in the expression for φ, ρ is replaced by $\rho-Div\ P$. The last named result is only what might have been expected in view of the fact that the $-Div\ P$ repesents the fictitious volume charge density.

These expressions by themselves would not be complete however. For in the process of building up the equations, it becomes implicitly involved that, on account of the displacements of the positive and negative electricity within the volume elements, there is left over at the bounding surface of the medium a charge which is everywhere equal to P_n, the outward normal component of P at the surface. In the case of a transition from one medium to another it is the discontinuity in P_n which is involved. Then, in a case of this kind, it is not in the spirit of a macroscopic analysis to regard a surface density of real charge as a limiting case of a volume density, but is preferable to leave it in the form of a surface density. On these accounts there will be two additions to the scalar potential, so that, using $\overline{P}_n$ to denote the discontinuity of P_n,

$$4\pi\varphi=\int\!\!\int\!\!\int\frac{[\rho-Div\ P]}{r}\,d\tau+\int\!\!\int\frac{[\sigma+\overline{P}_n]}{r}\,dS \tag{72}$$

Again, both the real and fictitious surface charge will contribute the equivalent of a surface current sheet, and both of these will consequently contribute to the vector potential. Furthermore, corresponding to the volume distribution of magnetization, there will be a surface distribution, which will also contribute to the vector potential. When all of these contributions are taken into account, the complete expression for the vector potential is found to be

$$4\pi c\,U=\int\!\!\int\!\!\int\frac{[j+\rho u+\frac{\partial P}{\partial t}+Curl\ [Pu]+c\ Curl\ m]}{r}\,d\tau \tag{73}$$
$$+\int\!\!\int\frac{[u\sigma+u\overline{P}_n+c\ Curl\ [n\overline{M}]}{r}\,dS$$

where the square brackets around the complete numerators of the integrals for φ and U indicate retarded potentials, but the other square brackets of course indicate vector products, and where $\overline{P}_n$ and $\overline{M}$ indicate the discontinuities of P_n and m at the boundaries. The n in the expression $Curl\ [n\overline{M}]$ refers to the unit vector in the direction of the normal.

Although, as we have already remarked, the origin and definition of the quantities P, E, B, etc., are not such that we can experimentally realize the quantities in terms of those definitions, nevertheless, having formed the above equations we may forget all about the origin of the expressions, and formulate the laws as follows:

Divide all space into elements of volume, and all surfaces separating different media into elements of surface. Then, it is possible to assign to each element of volume four numbers ρ, j, P, and m, the first two satisfying the relation

$$-\frac{\partial\rho}{\partial t}=Div\ (j+\rho u) \tag{74}$$

and to each element of surface, a number σ such that if φ and U be defined by equations (72) and (73) and E and B by the equations (69) and (70), the values of E and B so obtained for a point in free aether may be used in the force equation for the purpose of calculating the motion of any electron or charged body which can be handled by that equation.

The essential point of this formulation lies in the fact that no attempt is made to define the quantities P, E until after the truth of the law has been recognized, there is no measurement of forces within cavities or the like. Moreover, as regards the expression of what experiment can actually reveal, it would be possible to eliminate E and B from the discussion entirely in the manner indicated on page 28, for the case of the equations for free aether. It is an advantage to retain these quantities, however, not merely for analytical convenience, but also because it is in terms of them that the special properties of any particular medium are expressed. Thus in the above formulation, no statement is made as to any relation between such quantities as P and E other than those implied in the general equation. In practice, however, we know that P does depend upon E and possibly upon the other quantities B, u, etc. In the simplest case, of course $P=(K-1)E$. These constitutive relations, concerning which the general law tells us nothing are very important from the practical standpoint, since, if known, they limit the scope of generality in any particular problem, and serve to give enough relations to make the solution definite without the specification of more than a limited number of conditons. For the constitutive relations we must rely upon experiment in the case of any particular substance, although the restricted theory of relativity lends a valuable guide in requiring that, in the expression of these relations, the quantities related shall have the mathematical characteristics of being 4-vectors (see Appendix, Note 7.)

It may be of interest to conclude this part of the subject by giving a derivation of the Lorentzian equations for a material medium, by a process slightly different from that generally adopted, and one moreover following more closely the usual definition of polarization as commonly met with in the statical case.

A derivation of the Lorentzian equations for moving media.—As regards the determination of the field at a point outside of the medium, a point in a cavity for example, everything is known provided that we know the complete expressions for the scalar and vector potentials. It will be convenient to work with the non-retarded, or Maxwellian vector and scalar potential which we shall denote by A and ψ. It is then our purpose to determine the forms of A and ψ expressed in terms of *macroscopic* distributions, which shall have the same values as the A and ψ given by the sub-molecular analysis. Using subscript zero to denote quantities considered in the sub-molecular sense, and omitting the subscript when the quantities are to be understood in the macroscopic sense, we have

$$4\pi c A = \iiint \frac{(\dot{E}_0 + \rho_0 v_0)}{r_0} d\tau_0 \quad , \quad 4\pi\psi = \iiint \frac{\rho_0}{r_0} d\tau_0$$

The integrals are extended throughout all space, although, as regards contributions from points in free aether, there is no need to distinguish between a fine-grained and a macroscopic analysis.

Let ν be a macroscopic element of volume, and let Σ refer to summation taken throughout ν. Then

$$A = \frac{1}{4\pi c}\iiint \left\{\frac{1}{\nu}\Sigma \frac{(\dot{E}_0 + \rho_0 v_0)}{r_0} d\tau_0\right\} d\tau \tag{75}$$

where the integral is now taken with regard to the macroscopic elements $d\tau$, the subscript being omitted to denote the macroscopic nature of $d\tau$. The distinction between $d\tau$ and ν is merely formal, but $d\tau_0$ is still retained for the purpose of the summation denoted by Σ.[1]

If E only varied throughout the macroscopic volume element by quantities of the first order of macroscopic small quantities, we could write

$$\Sigma \frac{\dot{E}_0}{r_0} d\tau_0 = \frac{1}{r}\Sigma \dot{E}_0 d\tau_0$$

Since, in view of the sharp changes in charge density, this may not be the case, however, we must write

$$E_0 = E + \Delta E$$

where E is the true average value of E_0 throughout the volume element, and ΔE may be finite although $\Sigma \Delta E = O$. We thus have

$$\Sigma \frac{\dot{E}_0}{r_0} d\tau_0 = \frac{\nu \dot{E}}{r} + \Sigma \frac{\Delta \dot{E}}{r_0} d\tau_0$$

Thus, if u_x, u_y, u_z is the velocity of the matter, and if we separate off the parts j and ρu which are to be considered as conduction and convection current, we have

$$4\pi c A = \iiint \frac{(j + \rho u + \dot{E})}{r} d\tau + \iiint \left\{\frac{1}{\nu}\Sigma \frac{(\rho_0 v_0 + \Delta \dot{E})}{r_0} d\tau_0\right\} d\tau \tag{76}$$

where $\rho_0 v_0$ refers now only to the so called bound electricity.

Let us divide the electricity in the elements up into equal positive and negative parts, and let us associate these parts in pairs. The pairs may be taken in any way we please; but, for the purposes of a mental picture it will be convenient to think of corresponding elements of a doublet. Corresponding to each pair, we shall associate a point in ν which has an unvarying relation to the pair. Thus, it may be the point which moves so that it is always half way between the elements of a pair. It shall be the point to which we shall refer the position of the doublet, considered as a single entity.

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[1] The only reason for using Σ in place of integral signs is to avoid confusion between the summation throughout the element ν, and the summation or integration, in which ν itself figures as a volume element.

Let x, y, z, be the displacement of this point from the position which it occupied in the medium at some definite instant.

Let $\frac{1}{2}(\xi, \eta, \zeta)$ be the displacement of the positive element from it.

Let $-\frac{1}{2}(\xi, \eta, \zeta)$, be the displacement of the negative element from it, and let the corresponding dotted quantities refer to the corresponding velocities relative to the respective origins from which the displacements are measured. Then

$$\begin{aligned} 4\pi c A_x = & \iiint \frac{(j_x+\rho u_x+\dot{E}_x)}{r} d\tau \\ & + \iiint \left[\frac{1}{\nu} \Sigma \left\{ \frac{\epsilon(u_x+\dot{x}+\frac{1}{2}\dot{\xi})}{r_0} - \frac{\epsilon(u_x+\dot{x}-\frac{1}{2}\dot{\xi})}{r_0} + \frac{\Delta \dot{E}}{r_0} d\tau_0 \right\} \right] d\tau \end{aligned} \tag{77}$$

where ϵ here denotes the positive value of the element of charge, and u_x refers to the representative point for the pair, but r_0 refers, in each case, to the actual position of the charge. Thus,

$$\begin{aligned} 4\pi c A_x = & \iiint \frac{(j_x+\rho u_x+\dot{E}_x)}{r} d\tau \\ & + \iiint \frac{1}{\nu} \left[\Sigma u_x \epsilon \frac{\partial}{\partial s}\left(\frac{1}{r_0}\right) \delta s + \Sigma \frac{\dot{\xi}\epsilon}{r_0} + \Sigma \dot{x} \epsilon \frac{\partial}{\partial s}\left(\frac{1}{r_0}\right) \delta s + \Sigma \frac{\Delta \dot{E}}{r_0} d\tau_0 \right] d\tau \end{aligned} \tag{78}$$

where $\delta s^2 = \quad \xi^2+\eta^2+\zeta^2$

If p_x, p_y, p_z, refer to the component moments of a single pair, i.e. if

$p=\epsilon\times$(separation of the doublet elements)

$$\begin{aligned} u_x \epsilon \frac{\partial}{\partial s}\left(\frac{1}{r_0}\right) \delta s = & u_x \epsilon \left\{ \frac{\partial}{\partial x}\left(\frac{1}{r_0}\right) \xi + \frac{\partial}{\partial y}\left(\frac{1}{r_0}\right) \eta + \frac{\partial}{\partial z}\left(\frac{1}{r_0}\right) \zeta \right\} \\ = & p_x u_x \frac{\partial}{\partial x}\left(\frac{1}{r_0}\right) + p_y u_x \frac{\partial}{\partial y}\left(\frac{1}{r_0}\right) + p_z u_x \frac{\partial}{\partial z}\left(\frac{1}{r_0}\right) \end{aligned} \tag{79}$$

Since u_x and r_0 are constant to the first order of macroscopic small quantities throughout the volume ν, if we write

$$\nu(P_x, \ P_y, \ P_z) = (\Sigma p_x d\tau_0, \ \Sigma p_y d\tau_0, \ \Sigma p_z d\tau_0)$$

we have[1]

$$\frac{1}{\nu} \Sigma u_x \epsilon \frac{\partial}{\partial s}\left(\frac{1}{r_0}\right) \delta s = P_x u_x \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + P_y u_x \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + P_z u_x \frac{\partial}{\partial z}\left(\frac{1}{r}\right) \tag{80}$$

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[1] We may here parenthetically remark that, by considering the case u = constant, we see that the last two groups of three terms of (81), when integrated throughout the volume, represent respectively the contributions of the fictitious volume and surface charges to the vector potential. We might expect these to show themselves somewhere.

$$= -\frac{1}{r}\left\{\frac{\partial(P_x u_x)}{\partial x}+\frac{\partial(P_y u_x)}{\partial y}+\frac{\partial(P_z u_x)}{\partial z}\right\}$$
$$+\frac{\partial}{\partial x}\left(\frac{P_x u_x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{P_y u_x}{r}\right)+\frac{\partial}{\partial z}\left(\frac{P_z u_x}{r}\right) \tag{81}$$

Considering now the third member on the right hand side of (78), we have

$$\frac{1}{\nu}\Sigma\frac{\xi e}{r_0}=\frac{1}{\nu r}\Sigma\xi e=\frac{1}{r}\left[\frac{\partial P_x}{\partial t}+\frac{\partial}{\partial x}(P_x u_x)+\frac{\partial}{\partial y}(P_x u_y)+\frac{\partial}{\partial z}P_x u_z)\right] \tag{82}$$

The last three terms are added because the partial $\partial P_x/\partial t$ is equal to $\frac{1}{\nu}\Sigma\xi e$ (i.e. the rate of increase of moment per cc. in the *fixed* volume element, on account of the rate of separation of the doublet elements within it) plus the rate at which moment enters the fixed volume element on account of the motion of the medium.

Adding the results obtained in (81) and (82), and substituting in (78), we have

$$4\pi c A_x=\iiint\frac{j_x+\rho u_x+\dot E_x)}{r}d\tau+\iiint\frac{1}{r}\left\{\frac{\partial P_x}{\partial t}+\frac{\partial}{\partial y}(P_x u_y-P_y u_x)\right.$$
$$\left.-\frac{\partial}{\partial z}(P_z u_x-P_x u_z)\right\}d\tau+\iiint\left\{\frac{\partial}{\partial x}\left(\frac{P_x u_x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{P_y u_x}{r}\right)\right. \tag{83}$$
$$\left.+\frac{\partial}{\partial z}\left(\frac{P_z u_x}{r}\right)\right\}d\tau+\iiint\frac{1}{\nu}\left\{\Sigma\dot x e\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s+\Sigma\frac{\Delta\dot E_x d\tau_0}{r_0}\right\}d\tau$$

$$4\pi c A_x=\iiint\frac{(j_x+\rho u_x+\dot E_x)}{r}d\tau+\iiint\frac{\left\{\frac{\partial P_x}{\partial t}+curl_x[Pu]\right\}d\tau}{r}$$
$$+\iiint\left\{\frac{1}{\nu}\Sigma\dot x e\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s\right\}d\tau+\frac{1}{\nu}\iiint\left\{\Sigma\frac{\Delta\dot E_x}{r_0}d\tau_0\right\}d\tau \tag{84}$$
$$+\iiint\left\{\frac{\partial}{\partial x}\left(\frac{P_x u_x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{P_y u_x}{r}\right)+\frac{\partial}{\partial z}\left(\frac{P_z u_x}{r}\right)\right\}d\tau$$

The last term, when integrated throughout the volume leads, by Green's lemma to

$$\iint\frac{u_x P_n}{r}dS$$

this integral being taken over the surface of the medium. We thus see that, as regards the vector potential A_x at an external point, the medium acts as though there were

(1) A current density $j_x+\rho u_x+\frac{\partial E_x}{\partial t}+\frac{\partial P_x}{\partial t}+\{curl\ [Pu]\}_x$ throughout the volume.

(2) A current corresponding to the motion with velocity u_s of electric density P_n on the surface.

(3) An effect represented by the remaining terms of (84) which we shall presently discuss.

We may here remark that, as regards (1) we are in agreement with Lorentz, except that our polarization P has a meaning slightly different from his. We have defined the moment per cc. in terms of the summation of the moments of doublets by which the displacements have been artificially represented. The moment of each of these doublets has been obtained by multiplying its pole-strength by the *separation* between the poles. As already remarked, Lorentz considers the positive electricity present in some volume element, and multiplies its amount by say the x component of the distance through which it *has been* displaced. He then does the same thing for the negative electricity, and after adding the two results together, calls the sum P_x. In other words, the corresponding positive and negative portions which figure in the definition of P_x are, as it were, superposed on each other. His analysis appears to avoid the term

$$\int\int\int\left\{\frac{\partial}{\partial x}\left(\frac{P_x u_s}{r}\right)+\frac{\partial}{\partial y}\left(\frac{P_y u_s}{r}\right)+\frac{\partial}{\partial z}\left(\frac{P_z u_s}{r}\right)\right\}d\tau$$

occurring in (84), and he appears to obtain a current-density expression which is complete without it. This absence of the term is only apparent, however; for, in assigning different displacements to the positive and negative electricity in the volume elements, we are forced to have over, at the boundary of the medium, surface charges, if the displacements are to leave the medium uncharged as a whole. In a sense, we cannot average our current density throughout the macroscopic volume elements in such a way as to make the result the analytical equivalent of the fine-grained analysis, for, the effects which, in the fine-grained analysis, can be expressed in terms of the volume-element contributions alone can only be represented in the macroscopic analysis for the medium as a whole by contributions from the volume elements *together with* contributions from the surface. It may readily be shown that the surface charges which would be left over on the basis of the Lorentzian analysis would contribute by their motion exactly the term

$$\int\int\frac{u_s P_n}{r}dS$$

which comes from the application of Green's lemma to the last term of (84). In the case of a boundary between two material media it is of course the discontinuity in P_n which is involved.

It remains to consider the term

$$\int\int\int\frac{1}{v}\left\{\Sigma\dot{x}_e\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s+\Sigma\frac{\Delta\dot{E}_s}{r_0}d\tau_0\right\}d\tau$$

Now it readily follows from the definition of P that $-Div\ P+\rho=Div\ E$, i. e., $\rho=Div(P+E)$.
Thus

$$Div(j+\rho u)+Div\frac{\partial(E+P)}{\partial t}=Div(j+\rho u)+\frac{\partial\rho}{\partial t}=0$$

on account of the equation of continuity. Hence the divergence of

$$j+\rho u+\frac{\partial(E+P)}{\partial t}+Curl\ [Pu]$$

is zero, and the divergence of the corresponding portion of A is zero. Since the divergence of A as a whole is zero it follows that

$$Div\int\int\int\frac{1}{\nu}\left[\left\{\Sigma\Delta\dot{r}\epsilon\frac{\partial}{\partial s}\left(\frac{1}{r_0}\right)\delta s\right\}+\Sigma\frac{\Delta\dot{E}}{r_0}d\tau_0\right]d\tau=0$$

where $\dot{x}$ in the expression under discussion has been replaced by $\Delta\dot{r}$ to symbolize the resultant direction of the vector. Now the term under discussion represents a contribution to A as a result of electronic motions which exist when u is zero. Moreover, it is not part of the bodily motion ρu, nor of the conduction current j, nor of the effective current arising from the rate of change of the polarization. It represents, in fact, the part of A arising from the magnetization of the medium. It is this residual which we may *define* as the magnetization term.[1] Having said this much about it, we may avoid all further detailed consideration of the electronic motions responsible for it; and, remembering only that its divergence is zero, we may proceed as follows:
Whatever a may be, we have as a purely mathematical theorem:

$$4\pi a=-\int\int\int\frac{\nabla^2 a}{r}d\tau-\int\int\frac{1}{r}\left[\left(\frac{\partial a}{\partial n}\right)_1-\left(\frac{\partial a}{\partial n}\right)_2\right]dS$$

where the value of a on the left hand side corresponds to the value at a point in the region outside of the boundary of the medium (1), and the triple integral extends throughout all space. In our case, a is the contribution to the vector potential by the magnetization. Since $Div\ a=0$, we have

$$\nabla^2 a=-Curl\ Curl\ a=-Curl\ m$$

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[1] In the classical derivation of the formulae the $\Delta\dot{E}$ term does not appear to be considered. It does not show itself in the process of averaging the current density throughout the macroscopic volume element, although it should be made to do so. The percentage variations of E throughout the macroscopic volume element are of the same order of magnitude as those of the charge density ρ, and there is no more justification for neglecting their effects than there would be for saying that the electrostatic effect of a macroscopically neutral, but polarized volume element was zero because it contained equal quantities of positive and negative charge. Moreover, the services of the $\Delta\dot{E}$ term is required for the purpose of completing the magnetization term. Without it, this term would not give to the vector potential a contribution having zero divergence,and it would not correspond to what would be *measured* as the magnetization, even in the case of a non-moving medium.

where m is the value of the magnetization corresponding to the part a of the total vector potential. Thus,

$$4\pi a = \iiint \frac{Curl\ m}{r} d\tau - \iint \frac{1}{r}\left[\left(\frac{\partial a}{\partial n}\right)_1 - \left(\frac{\partial a}{\partial n}\right)_2\right] dS$$

Now it is easy to see that

$$-\left(\frac{\partial a}{\partial n}\right)_1 = [n_1 m_1],\ \left(\frac{\partial a}{\partial n}\right)_2 = -[n_2 m_2]$$

where n_1 and n_2 refer to the unit normals in the media (1) and (2) respectively. Thus, as far as the analytical contribution of the magnetization of the material medium to the vector potential is concerned, we have the equivalent of a distribution of current density of amount $c\ Curl\ m$ throughout the volume, and a surface current sheet of density $c[n_1 m_1]$ at the surface. In the case of a boundary between two material media it is the discontinuity of m which is involved.

Concerning the electrostatic potential ψ there is little to be said. The ordinary processes familiar in the development of Poisson's theory of dielectrics lead immediately to the fact that, as regards the polarization, the medium acts as though it were endowed with a fictitious volume charge of density $-Div\ P$ and a fictitious surface charge P_n per unit area.

Thus, to sum up, we have the following: Material media act as a whole as though there were

(1) a volume current density of amount

$$j + \rho u + \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + Curl\ [Pu] + c\ Curl\ m$$

(2) a volume charge density equal to $\rho - Div\ P$

But this specification is not sufficient, for we must have also

(3) a surface current density equal to

$$u\sigma + u\overline{P}_n + c\ Curl\ [n\overline{M}]$$

(4) a surface charge density equal to $\sigma + \overline{P}_n$

where $\overline{M}$ and $\overline{P}_n$ refer respectively to the discontinuities in m and P_n at the boundaries, and where σ represents the real charge density.

Moreover, it is not quite correct to say that (1) is confined to the volume elements and (3) to the surface. Thus, in the foregoing presentation, the term corresponding to uP_n occurred fundamentally in the volume elements. Although it is not there represented as an influence which is expressible in terms of *any* volume density of current macroscopically specified, it is of such a nature that, on integration throughout the volume, it contributes a result which is expressible in terms of a surface current.

In Lorentz's treatment, but with his special meaning for the polarization, (1) is completely representative of the action of the volume element as regards current. On the other hand, the surface terms such as uP_n are not avoided. This particular term appears now, not as a residue from the volume elements, collected up as it were by Green's lemma and deposited on the surface, but as a surface current resulting from the motion of the charges left over at the surface in order to preserve constancy of the total electric charge in the medium in spite of the displacements which constitute the polarization. Analogous remarks apply to the surface term resulting from the magnetization.

It will be observed that there is a fundamental difference between the representation of phenomena in terms of a fine-grained analysis and the representation in terms of the macroscopic analysis. For while in the former, the general law is expressible entirely by the differential equations, other matters such as boundary conditions applying only to the special problem, in the macroscopic analysis, the differential equation is not sufficient, and the surface effects play a part as sharing in the expression of the general law.

APPENDIX

Note 1. Concerning definitions in terms of a unit pole or unit magnet.—(Page 12). Thus, to illustrate a few of the difficulties involved in an endeavour to maintain logical precision in this matter, it may be remarked that the position of the axis of the unit magnet requires definition in terms of experiment. It is useless to say that it is the line joining the poles, without stating more explicitly what we mean by the statement. Perhaps the best that can be done is to say that if two similar magnets be turned so as to exert the greatest attractive force upon each other for a given distance, the line joining their centers is the axis of each. A small unit magnet may be defined in terms of the force which it exerts upon another just like it, when the two are placed with axes in line. A logical difficulty becomes involved, however, from a consideration of the alteration in moment which each produces in the other when the two are in each other's proximity. It is not safe to relegate this effect to the realm of trivialities; for, without further examination of the theory of the phenomena, one is unable to say whether the said inductive action is or is not a logical necessity of the actual existence of the magnetic state in either. Neither can one say that he will avoid concerning himself with these matters by *defining* the unit magnet in the above way and then sticking to his definition by calling it a unit magnet even after he has removed it from the influence of its neighbour. Truly, nobody can prevent one from defining a quantity in any way he chooses; but, we can readily see that a definition of unit magnet on the above lines and the subsequent definition of fields in terms of it would lead to fields whose properties are not those of which we have been accustomed to speak. The best that we can do in order to retain approximate precision in this matter is to include in our definitions a statement of how the moment of our test magnet is to be corrected on account of the field in which it exists and whose purpose it is to measure.

Again, one may say that he can approximate to a unit pole by taking a very long thin magnet. It is to be observed, however, that the proof that such a magnet acts in this way is something involving all sorts of hypotheses. We must first assume that magnetism is explicable in terms of doublets or their equivalents (amperian whirls for example) and then, after developing the mathematics showing that the action of the magnet can be represented in terms of fictitious volume and surface charges, an analysis involving the assumption of the law of inverse squares, moreover, we require the further assumption that our magnet is uniformly polarized before we can say that it can be represented entirely by a surface distribution of magnetism. When we have done all this we can show that if it is very thin, the surface distributions approximate to point singularities at the ends. One's mind rather revolts at the thought of putting the said pole inside an electron for the purpose of measuring the magnetic field there. Yet, electron theory does not hesitate to concern itself with the field within an electron.

Note 2. Several writers have given formulations of the laws of electro-magnetism in accordance with dynamical principles.

Equations for free aether.—(Page 23). Equations exactly corresponding to the form of the circuital relations of electromagnetic theory as applied to free aether were obtained by MacCullagh[1] as early as 1839 although without thought of their relation to electrical phenomena. Starting with the assumption that the kinetic energy was of the form

$$T=\frac{1}{2}A\int\int\int(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)d\tau \tag{85}$$

where ξ, η, ζ, corresponds to displacement of the medium, and assuming a purely rotational type of elasticity, so that the potential energy is of the form

$$V=\frac{1}{2}B\int\int\int\left[\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right)^2+\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right)^2\right]d\tau \tag{86}$$

MacCullagh by application of the Hamiltonian principle deduced the equations

$$-A(\ddot{\xi},\ \ddot{\eta},\ \ddot{\zeta},)=B\ Curl\ (f,\ g,\ h,) \tag{87}$$

where (f, g, h), is defined as regards its rate of change by

$$(\dot{f},\dot{g},\dot{h},)=Curl\ (\dot{\xi},\ \dot{\eta},\ \dot{\zeta}) \tag{88}$$

Equations (87) and (88) correspond exactly to the electromagnetic equations for free aether if aetherial velocity $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ is taken to represent magnetic field, and (f, g, h) representing curl of the aetherial displacement is taken to represent $\left(\frac{A}{B}\right)^{\frac{1}{2}}\times$ electric field. On this theory the kinetic energy is represented by

$$T=\frac{1}{2}\int\int\int(H_x^2+H_y^2+H_z^2)d\tau$$

Equations for media containing charges.—By utilizing the relation $H=Curl\ J$, where J has the meaning defined in equation 29, and by substituting in (89) and integrating by parts, the relation

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[1] "An Essay Towards a Dynamical Theory of Crystalline Reflection and Refraction." Trans. R. I. A., Vol. 21, Collected Works, p. 145.

$$\frac{1}{2}\int\int\int(H_x^2+H_y^2+H_z^2)d\tau=\frac{1}{2c}\int\int\int\Big\{J_x(\rho v_x+\dot{E}_x)+J_y(\rho v_y+\dot{E}_y)+J_z(\rho v_z+\dot{E}_z)\Big\}d\tau+\frac{1}{2}\int\int\Big\{(nH_y-mH_z)J_x+(lH_z-nH_x)J_y+(mH_x-lH_y)J_z\Big\}dS \qquad (89)$$

is obtained, the surface integral being extended over a boundary surrounding the region throughout which the volume integrals are taken. It has been customary to assume that the surface integral vanishes when the surface is taken at infinity, and that the two expressions

$$T=\frac{1}{2}\int\int\int(H_x^2+H_y^2+H_z^2)d\tau \qquad (90)$$

and

$$T=\frac{1}{2c}\int\int\int\Big\{J_x(\rho v_x+\dot{E}_x)+J_y(\rho v_y+\dot{E}_y)+J_z(\rho v_z+\dot{E}_z)\Big\}d\tau \qquad (91)$$

are equivalent.[1] The second form for T is the one which is used in deriving the equations for a medium containing electrons. As pointed out by Macdonald,[2] the neglect of the surface integral is in general not legitmate, and the two forms of dynamical theories which take respectively equations (90) and (91) for the kinetic energy are not the analytical equivalents of each other.

In view of the fact that it is customary in many branches of electromagnetic theory, to take the former of the above expressions as the appropriate one for kinetic energy, Livens has attempted to obtain the equations for a medium containing electrons on this basis.[3] Livens writes[4]

$$T=\frac{1}{2}\int\int\int H^2d\tau \text{ and } V=\frac{1}{2}\int\int\int E^2d\tau$$

He then carries out the variation of the Lagrangian function

$$\int dt\int\int\int(H^2-E^2)d\tau$$

subject to four conditions of constraint, viz.,

$$Div\ E-\rho=0, and\ Curl\ H-\frac{1}{c}\frac{\partial E}{\partial t}-\rho\frac{v}{c}=0$$

The second equation being a vector equation really expresses three conditions.

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[1] See J. Larmor, "Aether and Matter," p. 92

[2] H. M. Macdonald, "Electric Waves," p. 33.

[3] G. H. Livens, "The Theory of Electricity," pp. 567–572.

[4] We here make no distinction between B and H since we are concerned only with pure aether and electricity, and are not dealing with a macroscopic analysis.

The first of the above conditions of constraint introduces the undetermined multiplier φ, which subsequently turns out to be the electrostatic potential, while the other three conditions introduce undetermined multipliers which turn out to be the three components of the Maxwellian vector potential.

Although this analysis appears at first sight to avoid the difficulty inherent in the use of (90) for the kinetic energy expression, it must be observed that a surface integral becomes thrown out by the integration by parts occurring in the process of carrying out the variation of the Lagrangian function, and this surface integral involves the vector potential J. It is necessary to assume therefore that J vanishes to a higher order than $1/r^2$. In other words, in addition to the ordinary equations, the dynamical analysis automatically imposes this condition upon the systems which it is to regard as falling within the scope of its correlation. If we adopt this condition, however, the surface integral in (89) no longer gives any trouble. In other words it would appear that the difficulty under discussion is only avoided by the limitation of the problem to those cases in which it does not exist.

Considerable interest attaches to the manner in which the force equation evolves in the various dynamical analyses. According to the plan of the analysis sketched on pages (26) to (28) of this report, it would appear that the force equation should really take the form $E+\frac{[vH]}{c}=o$ at each point, even at a point within the electron. On the other hand Larmor appears to obtain relations of the type[1]

$$m\ddot{r}=E+\frac{[vH]}{c} \tag{92}$$

where the vector $\ddot{r}$ represents the acceleration, and m is a constant depending upon the size and constitution of the electron. In view of the customary expression of the equations of motion of an electron in the form "External Force" = Mass×acceleration, with the mass a function of the velocity, and also with the realization that even in this form the equation is only an approximation, it becomes of interest to trace the origin of (92).

The differential element of volume employed by Larmor is one containing a multitude of electrons. It is in fact a macroscopic volume element. Now, in building up the Lagrangian function, Larmor speaks of the "Part of the kinetic energy which involves the single electron moving with velocity $\dot{x}$, $\dot{y}$, $\dot{z}$," and he writes it in the form

$$\frac{1}{2}Le^2(\dot{x}^2+\dot{y}^2+\dot{z}^2)+e\frac{\dot{x}J_x}{c}+e\frac{\dot{y}J_y}{c}+e\frac{\dot{z}J_z}{c}$$

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[1] "Aether and Matter," p. 97.

where L is a constant. In writing the expression in this way, it becomes implicitly involved, as will be shown, that the quantity J which occurs in the expression and is instrumental in determing the vectors E and H which occur in (92) is not the whole of the vector potential, but only the part thereof which remains when the main contributions from the electron itself have been subtracted.

Expressed in vectorial notation, equation (28) for the kinetic energy takes the form

$$2cT = \int\int\int (J.\overline{\rho v + \dot{E}})d\tau$$

Let us consider all the contributions to this quantity, in which some one electron figures. The result is given by ΔT, where

$$c\Delta T = \underset{(1)}{\int\int\int}\left(\left\{\rho v + \dot{E}\right\}d\tau \underset{(2)}{\int\int\int}\left\{\frac{\rho v + \dot{E}}{r}\right\}d\tau\right)$$

where the integrals labeled (2) are taken throughout all space but r is the distance from the volume element to a point inside the chosen electron, and where the integrals labeled (1) are taken over the electron. We may divide the integrals (2) into two parts, a part which corresponds to the integral over the space of the electron, and a part which we shall call J_0, which corresponds to the integration over the remainder of space. We thus obtain

$$c\Delta T = \int\int\int\left(\left\{\rho v + \dot{E}\right\}d\tau \cdot \int\int\int\left\{\frac{\rho v + \dot{E}}{r}\right\}d\tau\right) + \int\int\int\left(\left\{\rho v + \dot{E}\right\} \cdot J_0\right)d\tau$$

Here the first pair of triple integrals are taken over the volume of the electron alone, while the third triple integral is taken throughout the space. Dissecting the expression still further, we have

$$\begin{aligned} c\Delta T = &\left(\int\int\int \rho v d\tau \cdot \int\int\int \frac{\rho v}{r} d\tau\right) + \left(\int\int\int \dot{E} d\tau \cdot \int\int\int \frac{\dot{E} d\tau}{r}\right) \\ &+ \left(2\int\int\int \dot{E} d\tau \cdot \int\int\int \frac{\rho v}{r} d\tau\right) + \left(J_0 \cdot \int\int\int \rho v d\tau\right) \qquad (93) \\ &+ \left(J_0 \cdot \int\int\int \dot{E} d\tau\right) \end{aligned}$$

where the taking of J_0 outside of the integrals can only be justified on the approximate assumption that, in so far as it is made up of contributions from regions other than that of the electron itself, its variation over this region is negligible. If we now write $v = v_0 + \Delta v$, where v_0 is the velocity of the center of the electron, and Δv is the departure from that value, (93) results in

$$\begin{aligned} c\Delta T = {} & (v_0\cdot v_0)\iiint\rho d\tau\iiint\frac{\rho d\tau}{r}+\left(\iiint\rho\Delta v d\tau\iiint\frac{\rho\Delta v}{r}d\tau\right) \\ & +\left(2v_0\iiint\rho d\tau\iiint\frac{\rho\Delta v}{r}d\tau\right) \\ & +\left(\iiint\dot{E}d\tau\iiint\frac{\dot{E}}{r}d\tau\right)+\left(J_0\iiint\dot{E}d\tau\right) \\ & +\left(2v_0\iiint\dot{E}d\tau\iiint\frac{\rho d\tau}{r}\right)+2\iiint\dot{E}d\tau\iiint\frac{\rho\Delta v}{r}d\tau \\ & +\left(J_0\cdot v_0\right)e+\left(J_0\cdot\iiint\rho\Delta v d\tau\right) \end{aligned} \tag{94}$$

Now as regards the terms on the right hand side, the first is obviously of the form $L_0e^2(v\cdot v)$, where the quantity L_0 is a constant depending upon the shape and distribution of charge in the electron. This term, together with the eighth member of the right hand side when written out, is responsible for the whole of the expression

$$\frac{1}{2}Le^2(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{e\dot{x}J_x}{c}+\frac{e\dot{y}J_y}{c}+\frac{e\dot{z}J_z}{c}$$

which is taken by Larmor as contributing to the portion of the kinetic energy in which the electron in question figures. As regards the remaining terms of (94) it will be observed that if all parts of the electron move with the same velocity, some of the terms vanish, leaving, however, the terms

$$\left(\iiint\dot{E}d\tau\iiint\frac{\dot{E}}{r}d\tau\right)+(J_0\iiint\dot{E}d\tau)+2v_0\iiint\dot{E}d\tau\iiint\frac{\rho}{r}d\tau$$

It is these terms which, even in the case of a rigid electron would play the major role in importing into the equation of motion those characteristics which are ordinarily depicted by a variation of mass with velocity.

In the form of the analysis as presented by Macdonald, the application is to true differential elements of volume as distinct from differential elements containing a large number of electrons. Macdonald includes, however, in his original expression for the Lagrangian function a part which he writes as $\iiint L'dxdydz$, and which follows right through the analysis and appears in the equation of motion in the manner given by

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial\dot{x}}\right)-\frac{\partial L'}{\partial x}-\rho\left(E_x+\frac{[vH]_x}{c}\right)=0$$

If L' were of the simple form $\rho_0\dot{r}^2/2$, this extra part of the Lagrangian function would of course simply result in

$$\rho_0\ddot{r}=E+\frac{[vH]}{c}$$

Although expressed in a somewhat different analytical form, the inclusion of a part L' in the Lagrangian function is the equivalent of the method adopted by H. A. Lorentz.[1]

In concluding these remarks upon the dynamical methods applicable to the subject it may be well to call attention to one additional matter. None of the dynamical developments appear to take account of that outstanding feature of the subject embodied in the fact that all electrons have the same charge. Would it not be possible to develop a more complete analysis in which this fact was taken into account in the form of an extra condition of constraint?

Note 3. Concerning the significance of v in the circuital relations.—(Page 25).—We may speak of the velocity of a point which is postulated to have some means of identification; but, to speak of the velocity in a continuous distribution of electricity is meaningless without further definition. When we speak of the velocity of a stream of water, we really mean the velocity of a cork in the stream or the average velocity of a molecule of which the stream is composed. Both of these definitions become without meaning as applied to the velocity of the electricity within an electron. If we are speaking of the velocity of a stream of electrons the situation is one of less perplexity; but the electromagnetic equations claim the power of applicability on an infinitely small scale, and, in some of their most important applications to the theory of electrons, the interrelation of the fields and motions in different parts of the same electron are matters of fundamental moment.[2] It is naturally of the greatest importance to understand what is to be meant by v in a case of this kind. Here again the question is largely one of the light in which we regard the definitions of the other quantities. We have already enlarged upon the difficulties in making logical definitions of E and H; and, without taking any responsibility for the definition of these quantities have set ourselves the task of inquiring how matters stand provided that we start from the hypothesis that there *exists* some logical definitions for them. Following out this line of thought, it would seem that equation (17) may be regarded in the following light.

E and H being assigned, define a quantity I by the relation

$$I = c\ Curl\ H - \dot{E} \tag{95}$$

where c is a perfectly arbitrary constant supposed at our disposal. It is of course only after c is assigned that our definition becomes explicit. Define v by the relation

$$v = \frac{I}{Div\ E} = \frac{I}{\rho} \tag{96}$$

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[1] "La Theorie Electromagnetique de Maxwell" "Archives Neerlandes des Sciences," v. 25, 1891–92.

[2] In calculating the so called "force which the electron exerts upon itself as a result of its motion," for example.

so that from (95) and (96),

$$\frac{1}{c}(\rho v+\dot{E})=Curl\ H \tag{97}$$

Now as regards the above definitions, nobody can prevent our defining I and v in any way we please. We cannot discover any laws of nature by definition, however; and, in general, the quantity v in (97) will have a very artificial meaning unless we make some further assumptions. To illustrate this, suppose we consider a spherical region (an electron for example) whose boundary is determined by the condition that *Div E* suffers a sharp change. Suppose, moreover, that this boundary is at rest. We might be inclined to say that the average value of v throughout the interior of the boundary is zero. How do we know, however, that electricity is not running in at the left side with infinite velocity, and out at the right hand side with infinite velocity, so that while the density outside of the electron is zero, ρv is finite there? Moreover, if we suppose a condition where the spherical boundary, as determined by the place where *Div E* suffers a sharp change, moves with a velocity u, then, quite apart from all question of relative motion of electricity within the electron itself, we are not sure that ρu represents the value of ρv which should occur in (97). For, we can imagine superposed on ρu a condition in which electricity runs into the boundary at the left and out of it at the right, with infinite velocity in such a manner as to contribute anything we like to ρv. It must be remarked that the above discussion in terms of infinitely small densities and infinite velocities, and the rather inexact use of words involved in speaking of them is only introduced here by way of illustration. In view of our purely artificial definition of I, it is conceivable that this quantity might have nothing whatever to do with the motion of electricity; in this case, however, our act of dividing it by a ρ which might have no existence, and so defining a v which when multiplied by the said ρ reproduced I would be obviously nothing more than a manipulation of symbols. Our desire is to emphasize the fact that, until some further assumption is made, there is no connection whatever between the velocity of the boundary of our charge and the v which occurs in equation (97). Co-existent with any set of motions which we like to assign to the boundaries of the charges, I, i. e., ρv can have any values whatever, as far as our definitions are concerned, and this is what might be expected in view of the fact that equation (97) figures merely as the definition of v. How then are we to import reality into the v of equation (97)?

Let us integrate the equation over a long period T

$$\int_0^T \rho v dt+\int_0^T \dot{E}dt=c\int_0^T Curl\ H\ dt \tag{98}$$

As in the above discussion, let us restrict ourselves for the time being to a case where ρ exists in the form of discrete particles (electrons), determined as regards their boundaries by the condition that *Div E* suffers a sharp change there. By taking the velocity which we could conceivably measure with a meter stick for any point on the boundary of one of the particles, we may, in the case where that velocity is in the same direction for some considerable time, speak of it as the velocity of the electron as a whole with an uncertainty of meaning which is, in the limit, infinitesimal, even though the electron changes in shape while in motion.[1] Let us, moreover, consider a case where the time integral of $\partial E/\partial t$ is negligible compared with that of ρv or of *Curl H*, so that we may write

$$\int_0^T \rho v dt = c\int_0^T Curl\ H\ dt \qquad (99)$$

Let us further integrate with respect to an element of area ds over an area of 1 sq. cm.

$$\int\int ds \int_0^T \rho v dt = c\int\int ds\int_0^T Curl\ H\ dt \qquad (100)$$

Without making any assumptions about the rigidity of an electron, let us consider an electron in the process of passing through the plane

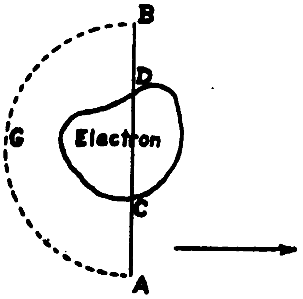

FIG. 2.

AB, Fig. 2. As a direct analytical consequence of equations (17) and (19), and without even the necessity of any detailed meaning to v as the velocity of electricity, it follows that

$$-\frac{\partial \rho}{\partial t} = Div(\rho v) \qquad (101)$$

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[1] In case one should argue that he has no means of fixing a point on the surface of the electron, we may suppose the velocity determined by taking the first reading for any one point, and the second reading for any other point. The difference of the readings divided by the time will, in the case of a long time interval, give the velocity with infinitesimal uncertainty as to its meaning.

and consequently

$$-\frac{\partial}{\partial t}\iiint \rho d\tau = -\iiint \frac{\partial \rho}{\partial t} d\tau = \iiint Div(\rho v) d\tau \tag{102}$$

the integrals being taken throughout any volume whatever. Applying Green's lemma[1] to the last member of (102), we have

$$-\frac{\partial}{\partial t}\iiint \rho d\tau = \iint \rho v_n ds = \frac{\partial}{\partial t}\iint ds \int_0^T \rho v_n dt \tag{103}$$

where the surface integral is taken over the surface surrounding the volume. Let us take our surface as composed of a portion AB large enough to include the greatest cross section of the electron as it passes through, and a surface such as the dotted surface, so that our complete surface of integration is ACDBGA.

Now it will be recalled that, in view of the artificial origin of ρv, a part of this quantity may represent a portion of I which has really no status in terms of motion of electricity, and which, by being forced into this form, calls for what would be described in physical terms as a continual flow of electricity *through* the moving electron, the velocity being infinite and the density zero at entrance and exit. If we wish to *postulate* that ρv_n is zero at all points where ρ is zero, the integral on the extreme right of (103) need only be extended over the plane AB, since the part over BGA will not contribute anything. Moreover, integrating over a time corresponding to that of the passage of the electron across the plane, we have, for the contribution of this electron to the integral on the extreme left hand side of (103) a value equal to the integral of $\rho d\tau$ throughout its volume, and this integral constitutes the definition of the charge on the electron. If N electrons pass through one square centimeter in the time T, the total value of the integral on the left hand side of (103) is Ne, so that, in view of (100),

$$Ne = c\iint ds \int_0^T Curl\ H\ dt .$$

where, in accordance with the above, the surface integral is taken over one square centimeter of the plane AB.

The assumption which has been made amounts to this, that it is possible to choose a constant c such that, with I defined as in (95), the whole of this quantity I may be represented as regards its time integral over a long period, by the transference of the charges on the electrons. Since

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[1] In order to apply Green's lemma in this form we must strictly have no discontinuity in $Div\ \rho v$. This difficulty is really present in most of the classical demonstrations where Green's lemma is applied to charges with sharp boundaries. It can be mathematically continuous, but as sharp as we please. With this understanding, our equation (103) holds.

$$\iint ds \int_0^T Curl\, H\, dt$$

is what is defined as the Heavisidean electromagnetic measure of the quantity of electricity passing through the square centimeter in the time T, while Ne is the Heavisidean electrostatic measure of this quantity, the ratio of the electromagnetic unit to the electrostatic unit is equal to c.[1]

It is important to realize that the element of *assumption* which has been here introduced, has been brought in not as a necessary part of the electromagnetic equations themselves, but rather in connection with their application to the particular problem of a stream of electrons. Equation (97), as it stands, involves no assumption; it is merely the definition of v, and is the only meaning to be attached to v at a *point* as distinct from the velocity of an electron as a whole. For the purpose of definition, c could be anything. The only place where assumption comes in is in discussing the contribution of those moving identifiable particles which we call electrons, and which we suppose to be characterized by showing a discontinuity, or at any rate a recognizably sharp change in ρ. It becomes of convenience to import into the definition implied in equation (97), the value of c which is chosen so as to fit the special problem in which assumption is involved, so that by the choice of this c, the v which is implicitly defined by (97) shall revert for that special case to its proper relationship with the meter-stick velocity of the electron as a whole.

Speaking in terms of physical intuition, but perhaps in terms which are not very exact, we may say that the significance of c is this: We may choose c to have any value we like, but then, in the case of a stream of electrons of constant velocity it will be necessary to regard the velocity of the electron as a whole as different from the average velocity of the charge in it. In other words, as we have before remarked, charge will flow in at one side of the electron and out at the other as the electron goes along. For any value of the velocity of the electron we can always choose a c such that the average v shall represent the meter-stick velocity of the electron; but, it is an assumption that the same c will serve in this respect for all velocities.

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[1] The exact specification of the conditions involved in speaking of the ratio of the electrostatic to the electromagnetic unit must necessarily involve a rather cumbersome presentation, if we use the electromagnetic unit only in the sense in which it is defined, viz., as the time integral of a current which is itself defined in terms of the curl of a magnetic field.

Note 4. Concerning definitions in terms of the motion of an electron.—(Page 31). If it be our intention to make definitions of that type which speaks of force on unit charge, unit pole, etc., the limit of refinement is reached when we choose as our entity on which to discuss the force the smallest particle whose existence we are willing to admit. Choosing then the electron as the entity in terms of which to construct our definitions, let us define the electric field E by the relation

$$E=k(\ddot{s})_{v=0} \tag{104}$$

where $(\ddot{s})_{v=0}$ is the acceleration which the electron would experience if it existed at the point in question with zero velocity, and, we should strictly add, with every time derivative of its coordinates, other than the acceleration, zero also. The constant k is to be regarded as perfectly arbitrary, and we shall find it possible to leave it so throughout, in such a manner as to avoid the difficulties accompanying the introduction of a material mass, the realization of a unit charge, etc.

As regards H, it will be realized that the fundamental physical idea involved in the usual definition in terms of the couple on a magnet, or current circuit, is the existence of a mechanical force on a current element. In its electronic aspect this reverts to the force on a moving electron. Using this as a suggestion to guide us, it might appear appropriate to *define* the component of H parallel to the axis of z for example by the equation[1]

$$k(\ddot{\xi})_{v\neq 0}=E_x+\frac{(v)_y H_z}{c}$$

where $(\ddot{\xi})$ represents the acceleration of the electron along the x axis, when the only velocity which it has is the velocity $(v)_y$ along the y axis. Recalling that $E_x=k(\ddot{\xi})_{v=0}$, the above would become

$$k(\ddot{\xi})_{v\neq 0}-k(\ddot{\xi})_{v=0}=(v)_y H_z/c \tag{105}$$

with corresponding definitions for H_x and H_y. As a matter of fact, we shall elaborate this definition slightly by expressing it in a form as typified by H_z as follows:

$$\frac{H_z}{c}=\mathop{\mathrm{L}t}_{v=0}\frac{k(\ddot{\xi})_{v\neq 0}-k(\ddot{\xi})_{v=0}}{(v)_y} \tag{106}$$

The reason for defining H in terms of (106) rather than in terms of (105) is this: We know that, in the language of ordinary electromagnetic theory, the mass varies with the velocity. In terms of the ordinary assumptions which occur in that theory, where (105) figures as the equation of motion of the electron, the quantity k on the left hand side of (104) would be multiplied by some function of the velocity. In other words, the acceleration $(\ddot{\xi})_{v \neq 0}$ is not independent of v. It depends only on v^2/c^2, however, whatever type of electron be assumed, so that H_z as defined by (106) approaches a definite limit which is independent of the different assumptions involved in the different electrons. Of course, nobody can prevent our *defining* H in any way we choose, for example, without the limiting condition specified in (106). If we do this, however, it will in general result that, in terms of the H we have so defined, we shall find equation (18) to be untrue.

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[1] It will at once be observed that this relation is suggested by $F=E+[vH]/c$. It would not, of course, be proper to attempt to define the magnetic field by supposing the electric field absent.

The ultimate form assumed by the circuital relations.—If, in terms of the definitions given by equations (104) and (105) we substitute in equation (18), the constant k cancels throughout, leaving the result in the form

$$-\underset{t_{(v_y=0)}}{\mathrm{L}} \frac{\frac{\partial}{\partial t}(\ddot{\xi})_{v \neq 0} - \frac{\partial}{\partial t}(\ddot{\xi})_{v=0}}{(v)_y} = \left(\frac{\partial \ddot{\eta}}{\partial x}\right)_{v=0} - \left(\frac{\partial \ddot{\xi}}{\partial y}\right)_{v=0}$$

with two other relations corresponding to the other two components. This involves nothing more than a statement of a relation between the derivatives of the motion of an electron at the point under different conditions. Not a single constant, not even c, is left.

On the basis of definitions which give E and H in terms of the motion of an electron as a whole, equation (17) can hardly be regarded as having a meaning except as applied in free aether, or in the hypothetical case of a non-electronic, and more or less continuous distribution of electricity. Viewed in this sense, however, and remembering that ρ is defined as *Div* E, we see again that when the vectors E and H are replaced by the quantities which define them, k cancels throughout, leaving an equation which expresses a relation between v and the derivatives of the motion of the electron at a point under different conditions.

With regard to this cancellation of k, the point is that although, to the extent that k is arbitrary, H is arbitrary, to that extent E is also arbitrary, and ρ is arbitrary; and, the relations between E, H, ρ, and v for one value of k are exactly the same as those between the corresponding quantities for any other value of k. In fact, the difficulty attending the conception of an ordinary material mass and a unit of charge in our definitions has disappeared as indeed it ought to. For, nobody ever observes a mass directly in the case of an electron . He writes down an equation with a mass in it, then does some experiments, calculates the mass, and finally puts it in a table in order that somebody else may insert it in his equations for the purpose of calculating how *his* observable electrons move. If we concern ourselves only with what is actually observed, we need not introduce the ideas of measure of charge or mass at all. Everything can be formulated in terms of a statement about how the time derivatives of the motion of the electron or other observable entity are related to each other under different conditions.

It must be remarked that the definition (106) does not constitute the complete equation of motion of the electron, since it is in the form of a limiting relation where the velocity is zero. This is in line with what might be expected, since we know that the complete calculation of the equation of motion of an electron involves some other assumption, such as that of Lorentz for example, which, in the ordinary language of electromagnetic theory gives the law in the form that the electron moves in such a manner that the total force which it exerts on itself due to its motion is equal and opposite to the force on it due to the external field.

Relation between the above definition of H and that ordinarily employed.—In the more coarse grained specifications employed in practise, it would appear that the most logical procedure is to first obtain the currents in the circuits in terms of the forces between the circuits in the manner outlined on pages 13-15. On this basis, the definition of H given by (8) is, in view of (9), the analytical equivalent of that which defines it at a point in terms of the mechanical force on the element of wire carrying the current at that point. As already pointed out, this method is applicable only in the case where the currents are steady or vary very slowly.

Now our equation (106) defines a quantity H which, in the case of an electron moving with small velocity results in an acceleration proportional to that velocity. If we assume that this action becomes transmitted to the wire, and finally to the spring which holds the coil, in the form of a pull proportional to the sum of accelerations which the electrons would suffer if free, the quantity H which we have defined by (106) falls into line with the more coarsely defined quantity referred to above, in the only case where that coarse-grained definition can be experimentally realized. Having defined the unit H in the coarse-grained way, in terms of our ordinary definitions, if we should now put a moving electron into the field and observe its acceleration, the constant k would of course become determined as the special value which caused the H defined by (106) to be numerically equal to the unit which is defined for the coarse-grained specification. But, for the purpose of the electromagnetic equations themselves, and for everything that they have to say about all that we can observe as regards the motions of electrons and so forth, correlations with the coarse-grained phenomena, with their attending approximations and vaguenesses of expression are unnecessary.

Note 5. The role played by definition, and by the force equation in relation to the invariance of the circuital relations under the transformation of the restricted theory of relativity.—(Page 35). It will be recalled that the application of the relativity transformation to the electromagnetic equations leads to a set of similar form but with

$$E_x,\ E_y,\ E_z,\ H_x,\ H_y,\ H_z,\ u_x,\ u_y,\ u_z,\ \rho$$

replaced by

$$E_x,\ \beta(E_y-\frac{v}{c}H_z),\ \beta(E_z+\frac{v}{c}H_y),\ H_x,\ \beta(H_y+\frac{v}{c}E_z),\ \beta(H_z-\frac{v}{c}E_y) \quad (107)$$

$$\frac{u_x-v,}{1-\frac{u_xv}{c^2}}\ ,\quad \frac{u_y}{\beta\left(1-\frac{u_xv}{c^2}\right)}\ ,\quad \frac{u_z}{\beta\left(1-\frac{u_xv}{c^2}\right)}\ ,\quad \beta\left(1-\frac{u_xv}{c^2}\right)\rho$$

$$\text{where } \beta=\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

Replacing the second set of ten quantities by the first set of ten letters, but with dashes, it is customary to say that the equations have reverted to the same form and are in accordance therefore with the restricted theory. This is really more than one is justified in saying until he has shown that the dashed quantities are really the quantities which the moving observer will measure for the fields, velocities and densities. The result follows for the velocities directly from the transformation equations themselves, but we cannot say anything about the fields until we have discussed the matter in terms of the definition of those quantities; and, it would seem that in the usual treatment of the subject, this question is dismissed too lightly. If we are to define the quantity E in the manner discussed in Note 4, i.e. in terms of the acceleration of an electron, we shall have E_y, for example, defined by the relation.

$$E_y=m_0\left(\frac{d^2y}{dt^2}\right)_0$$

where m_0 is an arbitrarily chosen number, and where the zero subscript against the parenthesis is to indicate that the electron considered has zero velocity, and, we must add, zero rate of change of acceleration and of all higher derivatives; for, without such specifications, as to the higher derivatives, the definition would be ambiguous. Now it may be shown directly from the relativity transformation that an electron which in the dashed system, has zero velocity, and acceleration $\ddot{y}'_0$ has, in the undashed system acceleration $\ddot{y}_v$, given by

$$\ddot{y}'_0=\beta^2\ddot{y}_v$$

Hence, if we postulate that the electromagnetic equations are invariant under the relativity transformation, we shall require the condition

$$\beta\left(E_y-\frac{v}{c}H_z\right)=m_0\beta^2\ddot{y}_v$$

Analagous remarks apply in the case of the transformation of the other components of E. The three equations corresponding to the three components are included in the general vector equation (49), which consequently represents a requirement imposed upon the equation of motion of the electron in order that the *circuital relations* may be regarded as invariant under the relativity transformation. It is customary to regard the invariance of the circuital relations as established before the force equation is discussed.

As far as the mere requirements of relativity are concerned, equation (49), since it represents an invariant relation,[1] could be true for the most general type of motion. It is important to realize, however, that for the purpose of providing for the invariance of the circuital relations, it is only *required* for the special case of an electron which has velocity, acceleration, and no higher derivatives.

It remains to discuss how H must be defined in order that the invariance of the circuital relations shall be provided for. To define it as the force on a unit pole, would lead us into untold troubles, including the variation of the mass of the unit pole with velocity. The only logical way of proceeding in line with the plan adopted in the case of E is to define H in the manner outlined in Note 4, i. e., by utilizing the force on the electron in a direction perpendicular to its velocity.

We may, in fact, crystallize the whole situation as regards the definition of E and H in the following manner:

As already remarked, it is a known fact that if we write down the relation

$$E+\frac{[uH]}{c}=\frac{d}{dt}\frac{ku}{\left(1-\frac{u_x^2+u_y^2+u_z^2}{c^2}\right)^{1/2}} \tag{108}$$

an application of the relativity transformation will lead to

$$E'+\frac{[u'H']}{c}=\frac{d}{dt'}\frac{ku'}{\left(1-\frac{u'^2_x+u'^2_y+u'^2_z}{c^2}\right)^{1/2}} \tag{109}$$

where x', y', z', t', u', refer to the transformed system, and E', and H', have the significance implied in (107). Now we may use (108) to define E and H in the following way: When the velocity of the electron is zero, (108) leads to

$$E=k(\dot{u})_{u=0}$$

which we may take as the definition of E, in line with the method adopted in Note 4. By putting $u'^2_y+u'^2_y+u'^2_z=0$, and consequently $u'=0$ in equation (109), we see that the same definition in the dashed system makes E' the electric field in that system.

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[1] The relation is invariant since, on multiplying throughout by $\left(1-\frac{u_x^2+u_y^2+u_z^2}{c^2}\right)^{-\frac{1}{2}}$ each side becomes converted into an expression capable of representing three of the four components of a 4-vector.

Again, if we write down the x component of equation (108) for the case where the velocity is entirely in the y direction, for example, we obtain

$$E_x+\frac{u_yH_z}{c}=\frac{d}{dt}\left[\frac{ku_x}{\left(1-\frac{u_x^2+y_y^2+u_z^2}{c^2}\right)^{1/2}}\right]_{u_x=u_z=0}$$

If we then replace E_x by $k(\dot{u}_x)_{u=0}$
and define H_z by the relation

$$\frac{H_z}{c}=\mathop{\mathrm{Lt}}\limits_{u_y=0}\frac{1}{u_y}\left[\frac{d}{dt}\frac{ku_x}{\left(1-\frac{u_x^2+u_y^2+u_z^2}{c^2}\right)^{1/2}}-k(\dot{u}_x)_{u_y=0}\right]_{u_x=u_z=0} \tag{110}$$

we shall have a definition which owing to the invariance in form of (108), defines H' in relation to the motions in the dashed system in the same way as H is defined in relation to the motions in the undashed system. In other words, the definition is one which permits of H' being regarded as the magnetic field in the dashed system.

By carrying out the limiting process as far as possible, the definition assumes the force

$$\frac{H_z}{c}=\mathop{\mathrm{Lt}}\limits_{u_y=0}\frac{k(\dot{u}_x)_{u_y\neq 0}-k(\dot{u}_x)_{u_y=0}}{u_y} \tag{111}$$

which corresponds to the definition (106) in Note 4. Similar remarks apply of course to the other components.

Case where E and H are not defined in terms of the motion of an electron.—If we wish to retain such a fine-grained specification of the equations that we may speak of the field at a point inside of an electron, definitions of the fields in terms of the motions of the electron as a whole are naturally too coarse-grained. It would now appear that the only logical procedure would be to follow the path outlined on pages (28) and (29), where E and H appear as subordinate quantities defined by equations (39) and (40), φ and U being in turn defined by (38), where the quantities ρ and v (which we shall here replace by u, to avoid confusion with the velocity of the system as a whole) are supposed determined on the basis outlined in the discussion of equations (36) and (37).

If we suppose the law typified by equations (36) and (37) to be restricted by the condition that it is invariant under the relativity transformation when that transformation is made subject to the condition that $\rho d\tau$ is invariant,[1] and that u transforms as a velocity, we shall have as follows:

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[1] An interesting case of this is where the law concerns itself entirely with the description of the motions of particles, and where the density simply represents the number of particles per c.c., and where any constants associated with the particles for the purpose of the description are invariant under the transformation.

Suppose that in terms of the invariant distributions $\rho d\tau$, two observers, A and A'', the former using x, y, z, t, and the latter using x', y', z', t', make corresponding definitions of φ and U and of φ'' and U''. Suppose that they then make, in the form of equations (39) and (40), but in their respective coordinate systems, corresponding definitions of E, H, and E'', H''. Then, by the very meaning of these quantities, E'' and H'' are the quantities which play the parts of the electric and magnetic fields for the observer who measures x', y', z', t'. It is, however, well known, and follows as an immediate consequence of the expressions (107), that

$$\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right) dxdydzdt=\left(\frac{\partial E'_x}{\partial x'}+\frac{\partial E'_y}{\partial y'}+\frac{\partial E'_z}{\partial z'}\right) dx'dy'dz'dt'$$

Hence, the quantity $\rho' dx'dy'dz'dt'$, with ρ' defined as *Div* E' is the same as the quantity $\rho dxdydzdt$ used by the observer A in the definition of φ and U, and consequently of E and H, and is therefore the same as $\rho'' dx'dy'dz'dt'$ used by the observer A''. Hence $\rho''=\rho'$. Since, however, relations of the form (38)–(40) determine uniquely the values of E and H for given values of ρ and u, and since it results directly from the equations of transformation that u' is the quantity which A'' measures for the velocity of a particle, it follows that the vectors E' and H' are everywhere equal to E'' and H'', and consequently represent the quantities which observer A'' would regard as the electric and magnetic fields.

It is interesting to observe that the sole criterion for our being able to define a set of relations which are invariant under the relativity transformation in the sense in which that statement has meaning, and which are of the form of the electromagnetic circuital relations is the following: That it shall be possible to formulate the laws of nature in terms of the motions of elements of volume in the manner sketched on page 32, the numbers N there referred to depicting some physical quantity such as charge within the element of volume, and that it shall be possible to do this in a form such that all observers using the same law shall arrive at the same values for the numbers in the elements of volume. The *usefulness*, or *fundamentality* of the relations will still depend upon whether or not the motions of a volume element can be described in terms of the vectors E and H at the point without explicit introduction of the motions and numbers attached to the other volume elements in space.

Note 6. (Page 43). *Calculation of the term*

$$W=\int\int\int \Delta v\left(E_1+\frac{[vH_1]}{c}\right)\rho d\tau$$

In the first place it is to be observed that the integral may be replaced by $\int\int\int \Delta v \, E_1 \rho d\tau$, since Δv and $[vH]$ are everywhere perpendicular to each other. If the velocity is in the positive direction of the axis of x,

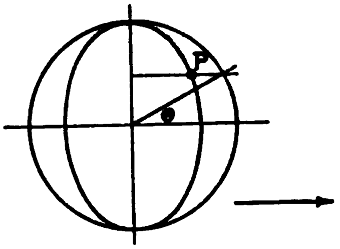

FIG. 3.

and if ξ is the perpendicular distance of a point P of the surface of the electron from the plane passing through the center of the electron and parallel to the yz plane, we have, for the Lorentzian electron,

$$\frac{\xi}{a \cos \theta} = \left(1 - \frac{v_0^2}{c^2}\right)^{\frac{1}{2}}$$

and

$$\Delta v = \dot{\xi} = -\left(1 - \frac{v_0^2}{c^2}\right)^{-\frac{1}{2}} \frac{v_0}{c^2} \cdot \dot{v}_0 \cdot a \, Cos \, \theta$$

Now, in the case of the Lorentzian electron, the elements of charge on the ellipsoid are obtained by projecting thereon, in a direction parallel to the axis of x, the corresponding elements of charge on the sphere which represents the electron in its un-contracted state. Thus, the contribution to the integral arising from the element of charge obtained by projecting the annular portion $2\pi a^2 Sin \, \theta d\theta$ of the sphere is

$$\Delta W = -\left(1 - \frac{v_0^2}{c^2}\right)^{-\frac{1}{2}} \frac{v_0 \cdot \dot{v}_0}{c^2} \cdot 2\pi a^3 \sigma \, Sin \, \theta \, Cos \, \theta \cdot E_1 d\theta \tag{112}$$

where $\sigma = e/4\pi a^2$ is the surface density of charge on the sphere.

Now the field E_1 consists of two parts, a part corresponding to the unaccelerated motion of the electron, and a part corresponding to the acceleration. The latter part contributes, to the integral, factors proportional to $\dot{v}_0$ and higher derivatives, so that, in virtue of the $\dot{v}_0$ terms already in (112), these terms give rise to contributions proportional to the square of the acceleration, or to contributions involving the product of the acceleration and higher time derivatives. The whole of the discussion on pages (41) to (43) is founded upon the acceptance of the quasi-stationary principle as a sufficient approximation, and to this degree of approximation we need not therefore concern ourselves with these terms, since the quasi-stationary principle cannot be expected to include this approximation. These terms contain within them the phenomena of radiation, which is neglected in the present discussion, but which has nothing to do with the difficulty immediately under consideration. Confining ourselves therefore in the present case to the portion of the field which does not involve the acceleration and higher derivatives, we recall that since the x components of the fields of the sphere and ellipsoid are the same at corresponding points, the value of E_x at the point P just outside the ellipsoid is $e \, Cos \, \theta / 4\pi a^2$. Just as, in the case of the system at rest, the average value of the field at a point within the charge on the surface is one half the value just outside, so here, as regards the integration across the thin element of charge at the point P, the effect is the same as if the magnitude of that element were multiplied by one half of the field just outside the ellipsoid at the point P. The value of W thus becomes

$$\begin{aligned} W &= -\frac{e^2}{16\pi a}\left(1-\frac{v_0^2}{c^2}\right)\frac{v_0\dot{v}_0}{c^2}\int_0^\pi Sin\,\theta\,Cos^2\,\theta d\theta \\ &= -\frac{e^2}{24\pi a}\left(1-\frac{v_0^2}{c^2}\right)\frac{v_0\dot{v}_0}{c^2} \end{aligned}$$

Note 7. (Page 47).—This would immediately suggest that it would be unlikely that we should find a suitable general relation between P and E without introducing, for example, the quantity B. To illustrate this matter we may remark that if we write $\epsilon=(1-q^2/c^2)^{-\frac{1}{2}}$, where q is the resultant velocity of a point, then it is known that, in the usual notation,[1]

$$\epsilon\left\{E+\frac{[uB]}{c},\ i\frac{(Eu)}{c}\right\}$$

$$\epsilon\left\{E-\frac{[uE]}{c},\ i\frac{(Bu)}{c}\right\}$$

$$\epsilon\left\{D+\frac{[uh]}{c},\ i\frac{(Du)}{c}\right\}$$

$$\epsilon\left\{h-\frac{[uD]}{c},\ i\frac{(hu)}{c}\right\}$$

are 4-vectors. Thus, one possible pair of relations between D, E, B, and H consistent with relativity would be

$$\left\{D+\frac{[uh]}{c},\ i\frac{(Du)}{c}\right\}=K\left\{E+\frac{[uB]}{c},\ i\frac{(Eu)}{c}\right\};\ \left\{B-\frac{[uh]}{c},\ i\frac{(Du)}{c}\right\}$$
$$=\mu\left\{h-\frac{[uD]}{c},\ i\frac{(hu)}{c}\right\}$$

where K and μ are constants. When u is small, these reduce to the well known approximations $D=KE$, and $B=\mu h$.

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[1] See, for example, E. Cunningham, "The Principle of Relativity," p. 119.

PART II.

UNIPOLAR INDUCTION

By JOHN T. TATE

I.

INTRODUCTION.

In his "Experimental Researches" Faraday[1] describes an experiment in which a current of electricity is induced in a wire whose ends are pressed against the surface of a cylindrical bar magnet spinning about its axis of symmetry. This type of induction was called "unipolar" by W. Weber[2] who analyzed experiments of this kind from the point of view of his two-fluid theory of magnetism. He regarded the induction as produced by the continuous passage of one pole only of a magnet through the current loop, the other pole remaining outside.

Today the term unipolar, acyclic, or homopolar is applied to the induction of an electric field in the vicinity of an axially symmetrical magnetic system which is rotating about its axis of symmetry. This magnetic system may be of any kind, permanent magnet, simple solenoid, or electromagnet. The problem of unipolar inducton has been correctly to account for the existence of this electric field. There seems always to have been an element of mystery connected with the phenomenon, and an element of doubt on the part of many physicists as to the power of electromagnetic theory to solve the problem uniquely.

It is proposed here to discuss the typical experiments on unipolar induction and their relation to modern electrical theory, to determine in how far the theory is in a position to predict the results of these experiments, and to what extent the results go beyond the theory in giving us information not already contained therein.

In the history of the development of the subject there has been a singular freedom from differences of opinion as to the experimental results, but at the same time a singular lack of agreement as to the way these results were to be interpreted. For that reason our attention in what follows will be directed primarily to the points of view of the different investigators rather than to details of experimental procedure. It will clarify the discussion, no doubt, if we first of all set forth briefly what present electromagnetic theory has to say about the unipolar induction problem.

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[1] Faraday, Experimental Researches, v. 1, p. 63.

[2] W. Weber, Pogg. Ann., 42, 1841.

II

ELECTRODYNAMICAL THEORY OF UNIPOLAR INDUCTION

A. *Fundamentals.*—The modern theory of electromagnetism proposes to calculate all effects from the positions and motions of charges. The attempt is made to correlate the motions of these charges in such a way that, given their positions and motions at any particular instant, it will be possible to predict the state of things at some later time. The mathematical framework about which the theory is constructed is usually put in the form of the field equations:

$$\begin{aligned} \rho u+\dot{E}&=c\ curl\ H \\ \dot{H}&=-c\ curl\ E \\ div\ E&=\rho \\ div\ H&=0 \end{aligned} \tag{1}$$

together with the so-called "force" equation:

$$F=E+\frac{[vH]}{c} \tag{2}$$

It will be unnecessary here to say anything about the significance of these relations or of the quantities entering into them. These matters are fully discussed in Part I of this report.

It is customary to integrate the above equations in terms of a scalar potential, ϕ, and a vector potential, A, where

$$\begin{aligned} \phi&=\frac{1}{4\pi}\int\int\int\frac{[\rho]}{r}d\tau \\ A&=\frac{1}{4\pi c}\int\int\int\frac{[\rho u]}{r}d\tau \end{aligned} \tag{3}$$

These potentials are subject to the restriction, equivalent to the equation of continuity for the charge, that

$$div\,A=-\frac{1}{c}\frac{\partial\phi}{\partial t} \tag{4}$$

In terms of these potentials, E and H are given by:

$$\begin{aligned} E&=-\frac{1}{c}\frac{\partial A}{\partial t}-grad\ \phi \\ H&=curl\ A \end{aligned} \tag{5}$$

The square brackets in (3) have the usual significance of indicating that in evaluating the integrals we are to take the values of ρ and of ρu at a time $\frac{r}{c}$ earlier than the time at which we wish the values of ϕ and A.

Substituting in (2) we have:

$$F=-\frac{1}{c}\frac{\partial A}{\partial t}-grad\ \phi+\frac{[vH]}{c}$$

or

$$F=-\frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int\int\int\frac{[\rho u]}{r}d\tau-\frac{1}{4\pi}grad\int\int\int\frac{[\rho]}{r}d\tau \\ +\frac{1}{4\pi c^2}\left[v,\ curl\int\int\int\frac{[\rho u]}{r}d\tau\right]. \tag{6}$$

In this form the electrodynamical problem has the appearance of being solved, for if we know the values of ρ and of ρu, we are in a position, through (6), to determine the force on a charge and hence its motion. It is not, however, easy, or indeed possible, to know under all circumstances what values to substitute in (6). This relation may, however, be regarded as a statement of our belief that it is always possible to assign values to ρ and ρu which satisfy the equation of continuity and which, when substituted in (6) will enable us correctly to calculate the motion of the charges.

B. *The basic problem of unipolar induction.*—The problem of unipolar induction is to account for the electric field in the vicinity of a rotating magnet. A magnet is a complicated structure made up of elementary magnets which are in some way connected with the atomic structure of the material of which the magnet is composed. Of the nature of the ultimate magnetic particle we know little or nothing. From the viewpoint of the theory sketched above it must be some form of current whirl of electricity, although the restriction there made that $div\ H=0$ may have been too drastic. It is entirely possible that true magnetic doublets exist. In any event, however, the unipolar induction problem, in the final analysis, reduces to the determination of the electric field in the vicinity of a moving elementary magnet.[1]

To solve the problem in the manner indicated in equation (6), above, requires that we know the values of ρ and of ρu for the moving magnet, assumed to be some sort of current whirl. But unfortunately we do not know their values even for the resting whirl. True we may postulate values which, when substituted in (5) will make E zero and give H the value appropriate to an elementary magnet, but that is not all that is necessary. It is essential that relation (6) be satisfied by the motion of the component parts of the whirl. These remarks may appear trivial since, after all, we do not care how the whirl is constructed so long as we can calculate its external effects. A little thought will show, however, that to calculate the field of the whirl after it has been set in motion, in terms of what it was before, necessarily involves a knowledge of how its structure is modified by the forces brought into play by its motion. To proceed requires that we bridge this gap in our knowledge by the introduction of an assumption not explicitly contained in the above theory. The test of the assumption will be the correctness of the results it predicts.

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[1] Swann, Phys. Rev., 15, 365, 1920, makes this problem the basis of his treatment of the theory of unipolar induction.

Our immediate inclination is to assign to the moving whirl a density $\rho'=\rho=O$ and a value of $\rho'u'=\rho(u+v)$ where ρ and u refer to the values for the resting whirl. If we do this and calculate the force on a charge at rest in the vicinity of the moving whirl, we arrive at a result which we regard as absurd, namely, that the charge will experience a force having a component in the direction of the motion of the whirl.

The safest way to proceed to the result is to give up the semblance of predicting the field of the moving whirl, and instead to assume the field suggested by the theory of relativity. We may then make use of (6) to find the values of ρ and of ρu which would have to be used in order to give that result. This is essentially the method used by Swann.[1] It turned out that, due to its motion, the density of the electricity of the whirl so changes that it becomes the equivalent, electrostatically, of an electric doublet of moment

$$N=\frac{[vM]}{c}$$

where M is the magnetic moment of the whirl. This means that in the force equation for the resting charge,

$$F=-\frac{1}{c}\frac{\partial A}{\partial t}-grad\ \phi$$

the scalar potential, ϕ, is not zero but is the electrostatic potential due to an electrical doublet of the above moment. Under these circumstances the force, F, will be

$$-\frac{[vH]}{c}$$

as is suggested by the theory of relativity.

2. The above method of solving the problem, by assuming the result, is a little unsatisfactory in that it seems a confession of weakness on the part of electromagnetic theory. It may be of interest to inquire to what extent the theory itself has the power to suggest the solution.

Let us suppose that the elementary magnet is made up of any system of charges in a stationary state of motion. Let the force on the electricity at any point of the system be

$$F_0=E+\frac{[uH]}{c}$$

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[1] Swann, loc. cit.

Let the system be set in uniform rectilinear motion with the velocity v, and assume that, at any rate for small values of v, the magnetic field and the force F_0 necessary to maintain stability of motion of the electricity, remain the same. Due to the motion of the system there will be an additional force, $\frac{[vH]}{c}$, acting on the electricity and hence to keep F_0 constant requires that a redistribution of electricity take place of such a nature that an electrostatic field $E' = -\frac{[vH]}{c}$ is created. If ρ_0 and ρ refer to the charge density in the system before and after being set in motion, we have:

$$\begin{aligned} \rho_0 &= div\ E \\ \rho &= div\left(E - \frac{[vH]}{c}\right) \\ &= \rho_0 - div\,\frac{[vH]}{c} \end{aligned}$$

and it is seen that the motion has produced an excess density of electricity

$$\Delta\rho = -div\,\frac{[vH]}{c} = \frac{1}{c}(v,\ curl\ H)$$

at each point of the current system.

We are now in a position to calculate that portion of the electric field in the vicinity of the moving system which is due to its motion. The scalar potential, ϕ, is given by:

$$\begin{aligned} \phi &= \frac{1}{4\pi}\int\!\!\int\!\!\int\frac{[\Delta\rho]}{r}d\tau = \frac{1}{4\pi c}\int\!\!\int\!\!\int\frac{[(v,\ curl\ H)]}{r}d\tau \\ &= \frac{1}{c}(v, A) \end{aligned}$$

where A is the vector potential of the system. Hence:

$$\begin{aligned} E &= -\frac{1}{c}\frac{\partial A}{\partial t} - grad\ \phi \\ &= -\frac{[vH]}{c} \end{aligned}$$

We therefore conclude that, in so far as the above assumptions are justified, any system of electric currents when in motion will, in virtue of a redistribution of its charges, exert a force on a resting charge in its vicinity given by:

$$E = -\frac{[vH]}{c}$$

where v is the velocity of the current system. This is a generalization of the statement made in the preceding section for the Amperian whirl. It is also applicable to a current system of finite size such as a solenoid.

3. *Field due to a true magnetic doublet in uniform motion.*—Admitting the possibility of the existence of true magnetic doublets, it is of interest to examine what the theory has to say about the effect their motion would have on an electric charge in their vicinity. To examine this point let us write the field equation in the symmetrical form:

$$\begin{aligned} \rho_e u_e + \dot{E} &= c\, curl\, H \\ \rho_m u_m + \dot{H} &= -c\, curl\, E \\ div\, E &= \rho_e \\ div\, H &= \rho_m \end{aligned} \tag{7}$$

Written in this way they enable us to take account of the possible existence of true magnetic doublets.

These equations may also be integrated in terms of potentials:

$$\begin{aligned} \phi_e &= \frac{1}{4\pi}\iiint \frac{[\rho_e]}{r} d\tau \\ A_e &= \frac{1}{4\pi c}\iiint \frac{[\rho_e u_e]}{r} d\tau \\ \phi_m &= \frac{1}{4\pi}\iiint \frac{[\rho_m]}{r} d\tau \\ A_m &= \frac{1}{4\pi c}\iiint \frac{[\rho_m u_m]}{r} d\tau \end{aligned} \tag{8}$$

which are subject to the restrictions,

$$\begin{aligned} div\, A_e &= -\frac{1}{c}\frac{\partial \phi_e}{\partial t} \\ div\, A_m &= -\frac{1}{c}\frac{\partial \phi_m}{\partial t} \end{aligned} \tag{9}$$

In terms of these potentials,

$$\begin{aligned} E &= -\frac{1}{c}\frac{\partial A_e}{\partial t} - grad\, \phi_e - curl\, A_m \\ H &= -\frac{1}{c}\frac{\partial A_m}{\partial t} - grad\, \phi_m + curl\, A_e \end{aligned} \tag{10}$$

The "force" equation has the same form as before except that here one must substitute the extended values for E and H given by (10). Thus:

$$F = -\frac{1}{c}\frac{\partial A_e}{\partial t} - grad\, \phi_e - curl\, A_m + \frac{[vH]}{c} \tag{11}$$

We may now examine the problem of finding the electric field in the vicinity of a true magnetic doublet in uniform motion. When the doublet is at rest we have:

$$\begin{aligned} &\phi_e=0\\ &A_e=0\\ &\phi_m=M_x\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+M_y\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+M_z\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\\ &A_m=0 \end{aligned}$$

When the doublet is set in motion with the uniform velocity, v, the obvious assumption to make is that

$$\phi'_e=\phi_e=0,\ A'_e=A_e=0,\ \phi'_m,=\phi_m,\ A'_m=\frac{v}{c}\phi_m$$

Let the motion be along the axis of x with the uniform velocity v. Then clearly:

$$\frac{\partial A'_m}{\partial t}=-v\frac{\partial A'_m}{\partial x}$$

Substituting these values in (10) gives:

$$\begin{aligned} &E'_x=0\\ &E'_y=-\frac{\partial A_m}{\partial z}=\frac{v}{c}H_z\\ &E'_z=\frac{\partial A_m}{\partial y}=-\frac{v}{c}H_y\\ &H'_x=-\frac{v^2}{c^2}\frac{\partial\phi_m}{\partial x}-\frac{\partial\phi_m}{\partial x}=\left(1+\frac{v^2}{c^2}\right)H_x\\ &H'_y=H_y\\ &H'_z=H_z \end{aligned} \qquad (12)$$

The electrical part of this field may readily be recognized as the field due to an electrostatic doublet, situated at the magnetic doublet with its axis perpendicular to the direction of motion and to the axis of the magnetic doublet, and of moment

$$N=\frac{[vM]}{c}$$

This is the same result as that obtained above, assuming that the elementary magnet was an Amperian whirl. The magnetic field of the doublet is unchanged by its motion except by terms of the order of $\frac{v^2}{c^2}$ It will be noted that on this view of things the term, $-curl\ A_m$, takes the part in the "force" equation (11) played by $-grad\ \phi$ on the other view. The scalar potential ϕ_e is here reserved for true electric charges.

As regards the question of the power of the above theory to predict the result, we may note that the solution depends upon the values which we assign to the quantities ρ'_e, $\rho'_e u'_e$, ρ'_m, and $\rho'_m u'_m$. The theory does not force any particular choice of these quantities upon us and in that sense has not the power to predict the result, at any rate in the present state of our knowledge.

4. *Rotation of a magnetic doublet about its own axis.*—We may now inquire what the theory has to say about the field in the vicinity of a magnetic doublet which is rotating about its own axis. In so far as it has the power to say anything it suggests that for the rotating doublet

$$\phi'_e = \phi_e = 0, \; A'_e = A_e = 0, \; \phi'_m = \phi_m, \; A'_m = A_m = 0$$

and thus that the field due to the rotating doublet is exactly the same as that due to the stationary one. If our elementary magnet is an Amperian whirl or any circular current system the same result is also strongly suggested as was pointed out by Pegram[1] for the case of a rotating solenoid and by Swann[2] for the Amperian whirl. Rotation of a circular current does not alter the differential velocity of the positive and negative electricity upon which alone the magnetic field depends.

It may be of interest in this connection to point out one way in which a current circuit is not equivalent, in respect of its external field, to a magnetic shell of uniform strength having the same boundary. For, as noted above, a circular current loop set in rotation about its axis would be surrounded by no electric field. A rotating magnetic shell, on the other hand, would be surrounded by an electric field since each elementary magnetic doublet of the shell would become, in addition, an electric doublet with axis radially out from the axis of rotation. There is thus in principle a fundamental difference between the unipolar induction effects of a rotating solenoid and those of a rotating material magnet.

As regards motion of translation, an electric current loop and a magnetic shell are equivalent in their effects.

To sum up the results of the last sections, it has been shown that, whatever the nature of the magnetic particle, or whatever the nature of the magnetic system, it will, when set in uniform translatory motion, be surrounded by an electric field which is, at any point, given by the expression

$$E = -\frac{[vH]}{c}$$

where v is the velocity of translation of the magnetic system. In this statement is contained the sole support which electromagnetic theory lends to the so-called moving line theory which regards the appearance of the electric field as due to the motion through space of lines of magnetic induction which are imagined rigidly attached to the magnetic system. It is to be especially noted that v refers solely to the translatory motion of the system and hence, in applying the moving line theory, the lines of magnetic induction must be regarded as partaking of the translatory motion of the system but not of the rotatory motion.

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[1] Pegram, Phys. Rev., 10, 591, 1917.

[2] Swann, *loc. cit.*

5. *The field surrounding a symmetrical magnet rotating about its axis of symmetry.*—In all unipolar induction experiments an essential feature is an axially symmetrical magnet rotating about its axis of symmetry. The theory sketched above will enable us to determine the field surrounding such a system. The electric field intensity, E, will be given at any point by:

$$E = -\frac{1}{c}\frac{\partial A_e}{\partial t} - grad\ \phi_e - curl\ A_m$$

Since the magnet is symmetrical and is rotating about its axis of symmetry the term $-\frac{1}{c}\frac{\partial A_e}{\partial t} = 0$. The term $-grad\ \phi_e$ is the ordinary electrostatic field due to the possible presence of electrical charges. The term $-curl\ A_m$ takes account of the field due to the motion of the magnetic doublets of which the magnet is composed. This latter field will be of a kind given by supposing that each magnetic doublet becomes, in addition, an electric doublet of moment

$$N = \frac{[vM]}{c} = \frac{\omega r M}{c}$$

where ω is the angular velocity of the magnet and r is the distance of the doublet from the axis of rotation. The net result for the whole magnet will be what we may term a fictitious polarization, P_0, of magnitude, at any point,

$$P_0 = \frac{\omega r I}{c} \tag{13}$$

The direction of this polarization will be in a plane through the axis of rotation and perpendicular to I.

A charge rotating with the magnet will experience an average force

$$F = -grad\ \phi_e - curl\ A_m + \frac{[vB]}{c}$$

where the last term results from its motion in the magnetic field whose true average value inside the magnet is B.

To determine what happens under this field it is necessary to know something about the nature of the material composing the magnet. If it be a dielectric the force acting on its charges will produce in it a true polarization of the ordinary type. Assuming the usual constitutive relation we may write this polarization as:

$$P = (\epsilon - 1)\ F = (\epsilon - 1)\left(-grad\ \phi_e - curl\ A_m + \frac{[vB]}{c}\right) \tag{14}$$

The field outside will be that due to the fictitious polarization, P_0, the true polarization, P, and whatever charges may be induced on conductors or dielectrics by these polarizations.

If the magnet be a conductor, electricity will move under the force, F, until, if possible, the force, F, is zero within the conductor. This will be possible provided F is derivable from a potential, and, given the symmetry of B, this is seen to be the case. Hence, if the magnet is a conductor, electricity will flow in it until at every point inside,

$$F=0=-\mathit{grad}\ \phi_e\ -\mathit{curl}\ A_m+\frac{[vB]}{c}$$

a relation which permits the calculation of ϕ_e.

The electric field strength, E, will be given by

$$E=-\mathit{grad}\ \phi_e-\ \mathit{curl} A_m,$$

It will be noted that that portion of $-\mathit{grad}\ \phi_e$ which cancelled $-\mathit{curl}\ A_m$ inside the magnet will do so outside as well, and hence:

$$E=-\mathit{grad}\ \phi'_e \tag{15}$$

where ϕ'_e is the electrostatic potential of the distribution of charge in the conducting magnet which is just necessary to cancel out, at points inside it, the motional intensity $\frac{[vB]}{c}$. Equation (15) contains the complete specification of the field due to a rotating symmetrical conducting magnet. Special cases may be solved explicitly. Swann[1] has worked out the case of a uniformly magnetized sphere and Rognley[2] that of a uniformly magnetized ellipsoid of revolution. It will be noted that in all cases the field is derivable from a potential and therefore will be completely screened off by any earthed shield surrounding the magnet but not partaking of its motion.

As a special application of equation (15) we may calculate the difference in potential between two points, 1 and 2, inside the magnet. We will have:

$$V_2-V_1=-\int_1^2 E_s ds=\int_1^2 \frac{[vB]_s}{c}\ ds$$

where the integral is to be taken over any line joining the two points. Since the field, E, is everywhere radial the two points and the path of integration joining them may, without loss of generality, be taken in the same meridian plane. Under these circumstances

$$[vB]_s=\omega r B_n$$

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[1] Swann, *loc. cit.*

[2] Rognley, Phys. Rev., 19, 609, 1922.

where B_n is the component of B lying in the meridian plane and perpendicular to ds, the other component being parallel to ds. We may therefore write:

$$V_2 - V_1 = \frac{1}{c}\int_1^2 rB_n ds = \frac{1}{2\pi c}\int\int B_n dS \tag{16}$$

where the surface integral is taken over the surface of revolution swept out by the path of integration in one revolution of the magnet. Clearly, since *div* $B = O$, we may write:

$$V_2 - V_1 = \frac{1}{2\pi c}\int\int B_n dS \tag{17}$$

where the surface integral is taken over *any* surface having as boundaries the two circles swept out by the points 1 and 2 in one revolution of the magnet.

It may be noted that equation (17) also gives the potential difference between two points in any symmetrical conductor, not necessarily magnetic, rotating in a magnetic field which is symmetrical about the axis of rotation. If the magnetic field is not symmetrical the field will not be derivable from a potential and eddy currents will be set up in the conductor.

III

UNIPOLAR INDUCTION EXPERIMENTS

A. *Experiments with closed circuits.*—The earlier experiments[1] differed but little in principle from Faraday's original experiment. The essential features in all of them were: (1) a symmetrical magnet, M, capable of being rotated about its axis of symmetry, and (2) a conducting circuit composed of two parts—one of which, A, rotated about the axis of the magnet, while the other, B, remained fixed. Brushes or mercury troughs maintained contact between A and B. Frequently the magnet itself formed the rotating part, A, of the circuit.

The experiments all showed that:—(1) an electromotive force is generated in the circuit by the rotation of A in the magnetic field, and this is true whether A is the magnet itself or not; (2) if A is separate from the magnet the electromotive force is entirely independent of whether the magnet is rotating or not. The integrated value of the electromotive force around the circuit was found in all cases to be:

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[1] For example, W. Weber, Pogg. Ann., 42, 1841; Pluecker, Pogg. Ann., 87, 351, 1852; Edlund, Ann. d. chim. et d. phys. (5) 16, 49, 1879; Exner and Czermack, Sitz. ber. d. k. Akad. d. Wissensch. zu Wien 94, 357, 1886; Budde, Wied. Ann., 30, 358, 1887; Hoppe, Wied. Ann., 28, 483, 1886; Hoppe, Wied. Ann., 29, 544, 1886; Hoppe, Wied. Ann., 32, 297, 1887; Lecher, Wien. Ber., 53, 1894; Wied. Ann., 54, 276, 1895; Wied Ann., 69, 781, 1899.

$$\frac{\omega}{2\pi c}\int\int B_n\,dS$$

where ω is the angular velocity of rotation of A and where the integral represents the flux of magnetic induction through any cylindrical surface having as boundaries the circles swept out around the axis of revolution by the two points of A which are in contact with B. As may be noted by comparison with equation (17) above, this result is predicted by the theory there developed.

The usual basis employed by the early investigators for the calculation of the electromotive force in these experiments was derived from the Faraday-Neumann law according to which the electromotive force around a closed circuit is proportional to the rate of diminution of the flux of magnetic induction through the circuit. As is well known, this integral law will be predicted by a theory which states that the electromotive force at any portion of the circuit is proportional to the rate at which that portion of the circuit is cutting across the lines of magnetic induction.

From the point of view of this theory, the lines of magnetic induction acquired at least enough of reality that one could speak of cutting across them and of their being carried along by the magnet. It is only natural, therefore, that in the unipolar induction experiment the question should have been raised as to whether, when the magnet rotates, they are carried around as though rigidly attached to it, or whether they remain stationary while the magnet rotates through them; in other words, whether the seat of the electromotive force is in the stationary or in the moving part of the circuit. Indeed, this question soon became practically the sole point of interest in the experiments, and many and varied were the modifications introduced in the hope of throwing light on this point. As Beer[1] and T. Preston[2] clearly pointed out, however, both hypotheses gave exactly the same result for the integrated value of the electromotive force around a closed conducting circuit and hence measurements depending on this integrated value were powerless to discriminate between them. For on the resting line hypothesis the rotating part of the circuit is cutting the lines of induction, and the electromotive force at any point is $\frac{[vB]}{c}$. Integrating this between any two points on the surface of the rotating conductor gives, (*cf.* equation (17)):

$$\int_1^2 F_s ds = \frac{\omega}{2\pi c}\int\int B_n\,dS$$

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[1] Beer, Pogg. Ann., 87, 351, 1852.

[2] T. Preston, Phil. Mag., (5), 19, 131, 1885.

where the surface integral has the same significance as before. The resting part of the circuit will contribute nothing to the integral and hence this is the total electromotive force around the circuit. If, on the other hand, the lines of force rotate with the magnet both the moving and the fixed parts of the circuit will contribute to the integrated value of the electromotive force, except in the special case where the rotating part of the circuit is rotating with the same angular velocity as the magnet. But a simple calculation will show that the result for the whole circuit is exactly the same as on the other hypothesis.

Both hypotheses had strong adherents,—for the stationary lines, Faraday, Pluecker, Lecher; for the rotating lines, Weber, Preston, Hertz, Lodge, Rayleigh. Faraday[1] very clearly stated his opinion in these words:—"When lines of force are spoken of as crossing a circuit, it must be considered as effected by the translation of a magnet. No mere rotation of a bar magnet on its axis produces any inductive effects on circuits exterior to it. The system of power about the magnet must not be considered as rotating with the magnet any more than the rays of light which emanate from the sun are supposed to revolve with the sun. The magnet may even in certain cases, be considered as revolving amongst its own forces, and producing a full electric effect sensible at the galvanometer." It is interesting to note that with a little more precision of statement, in which one would speak of the fields of the individual magnetic doublets of which the magnet is composed rather than of the field of the magnet as a whole, the preceding might very well have been written at the present time. It expresses about as well as could have been expressed at that time the present view of the moving line theory.

Clearly, though, the arguments advanced from either view point could be little more than expressions of opinion until some way of experimentally differentiating the two hypotheses could be devised.

A number of investigators had from time to time expressed the belief that an electrostatic examination of the field surrounding the rotating magnet would discriminate between the two views. T. Preston[2] proposed such an experiment and Hertz agreed that the results would be capable of deciding the matter. Unfortunately the experimental facilities available at that time were wholly inadequate for the success of the measurements. Lecher[3] attempted some electrostatic experiments but without success.

About 1900 another view began to be in evidence. Poincaré[4] and Abraham[5] both dismissed the whole question of the rotation or non-rotation of the lines of induction by saying it had no meaning. Poincaré attempted to demonstrate this by pointing out that for closed circuits both hypotheses agreed in giving correct results but that for open circuits both agreed in giving wrong results. He discusses the theory of unipolar induction from the energy standpoint, but the expression used in finding the electromotive force around a closed circuit is the analytical equivalent of assuming it to be:

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[1] Faraday, Phil. Trans., 1852, p. 31.

[2] T. Preston, Phil. Mag., (5), 31, 100, 1891.

[3] Lecher, *loc. cit.*

[4] Poincaré, L'Eclairage Electrique, 23, 41, 1900.

[5] Abraham, Theorie d. Elektrizitaet, Vol. 1, p. 418.

$$\int \frac{[vB]}{c} ds \tag{18}$$

exactly as before. This expression, as we have seen above, is true in general for a conducting circuit only. Nevertheless, in discussing a proposed experiment on open circuits, an experiment essentially the same as those afterward carried out by Blondlot[1], Wilson[2], and Barnett[3], Poincaré uses the expression in calculating the difference in potential between two points on the surface of an ebonite ring rotating in a magnetic field. Use of expression (18) for dielectrics is the characteristic feature of Hertz' theory of moving media and leads to an incorrect result as shown later by the experiments above mentioned. In discussing the light which open circuit experiments would throw upon the question of whether or not the lines of induction rotate, he adopts, for purposes of criticism, an unusual moving line theory according to which dielectrics are completely inert to the motion through them of lines of induction. It is not to be wondered at that he dismissed such a theory as meaningless.

B. *Electrostatic experiments.*—On the experimental side the matter rested here until about 1912 when a series of experiments by Kennard[4] and Barnett[5] were reported. These were experiments in which the displacement of electricity on a conductor in the neighborhood of a rotating magnet or solenoid was looked for. The experimental arrangements in all of them were very similar. The essential features were: (1) a magnet or solenoid mounted to rotate about the axis of symmetry of its magnetic field, and (2) a coaxial cylindrical condenser sometimes also mounted to rotate. In all of them the rotating magnet was screened from the condenser by an earthed shield. The purpose of this was to shield the condenser from spurious effects and at the same time to eliminate any effect due to the electrostatic field which, on the resting line theory, would surround the rotating magnet.

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[1] Blondlot, Jour. de Phys., Jan., 1902.

[2] Wilson, H. A., Roy. Soc. Trans., A, Sept., 1904.

[3] Barnett, Phys. Rev., 27, 425, 1908.

[4] Kennard, Phil. Mag., 23, 937, 1912.
Phys. Zt., 13, 1155, 1912, and 14, 250, 1913.
Phys. Rev., 1, 355, 1913.
Phil. Mag., 33, 179, 1917.

[5] Barnett, Phys. Zt., 13, 803, 1912; 14, 251, 1913.
Phys. Rev., 35, 323, 1912.

The usual experimental procedure was as follows: (1) set the magnet in rotation, (2) connect the two cylinders of the condenser momentarily by a wire, (3) break the connection, (4) stop the rotation of the magnet, and (5) test the insulated cylinder for charge. The results all agreed in showing no effect comparable with that which would have been obtained had the cylindrical condenser been rotated in the field of the magnet.

On the basis of the theory developed above, no effect should have been obtained since, according to it, the field surrounding a rotating magnet is due entirely to a distribution of charge throughout the rotating magnet and would be completely screened from the condenser by the earthed shield surrounding the magnet.

We may now consider the relation between the negative results of these experiments and the vexatious question of whether or not the lines of magnetic induction rotate with the magnet. Kennard and Barnett differed completely in their interpretation of the results. Kennard concluded that here was definite evidence that the lines of induction did not rotate with the magnet, whereas Barnett argued that the experiments had nothing to say on this point. It is always very difficult to be sure that one understands correctly the view point of others and especially is this true in the field of electromagnetic theory into which so many symbols and names of things are introduced and so few defined. In the writer's opinion, however, the difference between the conclusions of these investigators is due primarily to a difference in what they have agreed to mean by the rotating line theory. Kennard's view, and perhaps the usual one, is that the rotating line theory requires that the force on a resting charge in the vicinity of an uncharged rotating magnet be given by an expression

$$F = -\frac{[vB]}{c}$$

where v is the linear velocity, at the charge, of a line of induction which is imagined fixed rigidly to the magnet and rotating with it. On this view the results of the above experiments definitely decide against the theory.

Barnett argued, however, that the expression for the force must be modified to include a term, say E, which would express the force on the charge due to a "displacement" in the aether brought about by the motion of the lines of induction through it. This displacement in the aether is assumed to be of just the right amount to cancel completely the force on the charge; that is, $E = \frac{[vB]}{c}$. If one is at liberty to assume that this field, E, is due to a fictitious distribution of electricity in the aether, the density of that distribution would be *div* E. Regarding the aether as a dielectric, this would mean a polarization of magnitude such that $-div\ P = div\ E = div\ \frac{[vB]}{c}$. That is, the polarization in the aether would be proportional to the motional intensity;

$$P = -\frac{[vB]}{c}$$

Toward the field due to the moving lines of induction it would therefore appear that the aether is assumed to be a dielectric of infinite specific inductive capacity. On the other hand it is necessary to assume that as regards fields of the ordinary electrostatic type it is a dielectric of unit specific inductive capacity, otherwise it would be impossible to exert any force on a charge imbedded in it.

With this view of matters one must be a little careful in considering the problem of the force on a charge due to the *rectilinear* motion of a magnetic doublet. Presumably the aether will react to the motion of the lines of induction which are carried along with the rectilinear motion of the doublet in the same way as it does to their rotation. The effect in this case would be also to annul all forces on the charge. To do this the aetherial polarization would be such as to deposit on the moving doublet a fictitious electric doublet of moment $-\frac{[vM]}{c}$. We know, however, that in this case there is a resultant force on the charge and it is therefore necessary to make some additional hypothesis. The most natural one to make is that, due to its motion, the magnetic doublet becomes in addition an electrical doublet of moment $\frac{[vM]}{c}$, in exactly the sense of the theory we have developed above. There can be no fundamental objection to this view of things but the expediency of introducing such an hypothetical aetherial polarization might be questioned. It would appear that unless there is some justification for it elsewhere in electromagnetic theory it would be purposeless to retain it here for the sake of a moving line theory.

Since he had no hope of discriminating between the two hypotheses as to the motion of the lines of induction, Barnett's chief interest in the experiments lay in the fact that in them the effect did not depend on the relative motion of the material parts of the system. As he points out, when the magnet is at rest and the condenser rotating an effect is obtained, whereas if the condenser is at rest and the magnet rotating none is obtained. There would seem to be no reason, a priori, for expecting the results to be the same, however, since the relative motions, in the general sense of the word, are by no means the same.

Recently Barnett[1] described an experiment in which he hoped to obtain evidence as to the existence or non-existence of the aether. Presumably the term aether is here applied to a medium of some such properties as those discussed above. In this experiment an electromagnet was mounted on a long pendulum support in such a way that its pole pieces, one on either side, could be swung past an earthed metal box containing a plane plate condenser with plates parallel to the line of motion of the magnet and to the magnetic field. The size of the pole pieces relative to the condenser was such that, at the moment when the magnet was in the center of its swing, the condenser and box might, without serious error, be regarded as at rest in the uniform field between two magnetic poles moving with a uniform velocity. The two poles of a similar but stationary electromagnet were brought into such a position that at the same moment the plane of demarkation between the lines of force from the two magnets, with poles in opposition, was coincident with the upper surface of the lower condenser plate. In this way Barnett hoped to screen the portion of the box below the lower condenser plate from the inductive effects produced by the moving magnet. Barnett[1] later expressed doubt as to the validity of this assumption; and its weakness is apparent. Under any circumstances, however, it is clear, as pointed out by Swann[2], that the earthed metal box surrounding the condenser would completely shield it from any electric field due to the moving magnet. The experimental results were entirely negative; no charging up of the condenser was observed. Barnett concluded that this negative result pointed to the existence of the aether and that it was inconsistent with the theory of relativity. That this conclusion was unjustified is evidenced by the fact that a negative result is predicted by a theory perfectly consistent with the theory of relativity.

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[1] Barnett, Phys. Rev., 12, 95, 1918.

In 1917 Pegram[3] completed a series of careful experiments essentially similar to those of Kennard and Barnett, in which a cylindrical condenser inside a rotating coaxial solenoid could be maintained at rest or rotated with the solenoid. With both solenoid and condenser rotating he obtained the charging up to be expected. With the solenoid rotating and the condenser at rest no effect was obtained, completely confirming the previous experiments.

In discussing the theory of unipolar induction Pegram emphasises the importance of the force equation in the form:

$$F=-\frac{1}{c}\frac{\partial A}{\partial t}-grad\ \phi+\frac{[vB]}{c}$$

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[1] Barnett, Report on Electromagnetic Induction, A.I.E.E., Oct. 10, 1919. Phys. Rev., 15, 527, 1920.

[2] Swann, *loc. cit.*

[3] Pegram, Phys. Rev., 10, 591, 1917.

and in particular the importance of the first term $-\frac{1}{c}\frac{\partial A}{\partial t}$, involving the time rate of change of the vector potential. On the view of the theory given in II B 1 and 2, above, it would appear that in unipolar induction experiments it is rather the term $-grad\ \phi$ which is of greater importance, indeed in many cases the term $-\frac{1}{c}\frac{\partial A}{\partial t}$ is zero. For example, in one of the instances cited by Pegram, a charge finds itself inside two coaxial solenoids in which the currents are so adjusted that everywhere inside the inner solenoid the magnetic field is zero. The inner solenoid is then displaced transversely and the charge experiences a force. It would appear that in this case the term $-\frac{1}{c}\frac{\partial A}{\partial t}$ is zero and that the sole contribution to F is the term $-grad\ \phi$, which arises from the redistribution of charges in the wires of the solenoid discussed in II B 2 above.

Pegram clearly points out that on the crudest view of the electron theory of conduction it would be improbable that a solenoid rotating about its axis could exert a force on an electric charge in its vicinity and refers to Larmor[1] and Howe[2] who had also come to that conclusion. It is for this reason that there is an essential difference, not always realized, between unipolar induction experiments done with rotating solenoids and those done with rotating material magnets.

Recently Swann[3], who had previously[4] given an extremely lucid analysis of the whole theory of unipolar induction from the point of view of electrodynamical theory, reported on an experiment in which an effect due to the charging up of a sphere of iron rotating in a uniform magnetic field was measured. The axis of the rotating sphere was earthed and the charge which was induced on a metal shield completely surrounding the rotating sphere was measured. Considering the difficulties associated with measurements of that kind the results were remarkably consistent and were in complete agreement with the theory as outlined above.

C. *Status of the moving line theory.*—In all the experiments discussed thus far the experimental arrangement has been such that at most all that could be tested was whether the effects could be attributed to the usual view of the rotating line theory. They have decided against that view. It seems to have been usually assumed that there were but two possibilities, either the lines of induction rotate or they stand still. As first pointed out by Swann, however, and as is evident from

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[1] Larmor, Roy. Soc. Phil. Trans. A, 727, 1895.
[2] G. Howe, Electrician, 76, 169, 1915.
[3] Swann, Phys. Rev., 19, 38, 1922.
[4] Swann, *loc. cit.*, ante.

II B 4 above, neither one of these views is in accord with electromagnetic theory. In order to be consistent with this theory it is necessary, if one wishes to think in terms of moving lines of induction, to imagine the lines of induction from each elementary magnetic doublet or Amperian whirl of which the magnet as a whole is composed, as carried along in the translatory motion of the doublet but not in any way partaking of its rotatory motion. Swann goes a step further and shows that if an elementary magnetic doublet is rotating about any axis making an angle with its own, that then one must regard the lines of force emanating from the two fictitious poles of the doublet as partaking of the translatory part of their motion only and not of their rotatory part.

It is interesting to inquire in how far this view of the motion of the lines of induction is verified by experiment. As pointed out above, the experiments of Kennard, Barnett, and Pegram indicated that the lines of induction did not partake of the rotation of the magnet as a whole. Is there any experiment in which, to account for the results, one must take account of the motion of translation of the elementary magnets composing the magnet? Clearly such an experiment would have to be one in which the rotating magnet is a non-conductor for, as pointed out above, if the magnet is a conductor the field due to the fictitious polarization produced by the individual translatory motions of the elementary magnets is completely cancelled out both at points inside and at points outside the magnet. In the case of the conducting magnet, therefore, the hypothesis of the stationary lines of induction will yield correct results. But this would no longer be the case if the magnet were a non-conductor.

The importance of doing an experiment of this kind on a non-conducting magnet was first pointed out by Einstein and Laub[1]. These authors showed from relativity considerations that if in the Wilson experiment, in which a cylindrical dielectric is rotated in a magnetic field parallel to its axis, the dielectric were at the same time magnetic the effect observed (for instance, the potential difference between the inner and outer coating of the dielectric) should involve the factor $\frac{\epsilon\mu-1}{\epsilon}$ instead of the factor $\frac{\mu(\epsilon-1)}{\epsilon}$ which they ascribed to the theory of Lorentz.

We may show, however, that a proper application of the theory of Lorentz to an experiment of this kind will not lead to an effect proportional to $\frac{\mu(\epsilon-1)}{\epsilon}$ but to $\frac{\epsilon\mu-1}{\epsilon}$ as suggested by relativity considerations. Let us consider for simplicity a magnetic dielectric in the form of a hollow cylinder rotating with uniform angular velocity, ω, about its axis in a uniform magnetic field parallel to the axis of rotation. Let I be the intensity of magnetization and suppose, as is customary, that

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[1] Einstein and Laub, Ann. d. Phys., 26, 532, 1908.

$$I=(\mu-1)H$$

where μ is the permeability of the dielectric. When the dielectric is set in rotation a fictitious polarization

$$P_0=\frac{\omega rI}{c}$$

will appear in conformity with the ideas developed above. This polarization is directed radially out from the axis of rotation and the electrostatic field due to it is clearly $-P_0$. This field, together with the motional intensity $\frac{\omega rB}{c}$, acting on a charge rotating with the dielectric will conspire to produce in the dielectric a true polarization, P. Since both the field $-P_0$ and $\frac{\omega rB}{c}$ are radial the resulting polarization will be radial and the electrostatic field due to it will likewise be $-P$. In equilibrium we shall have, assuming that there are no other charges anywhere,

$$\begin{aligned} P&=(\epsilon-1)\left(E+\frac{\omega rB}{c}\right)\\ &=(\epsilon-1)\left(-P-P_0+\frac{\omega rB}{c}\right) \end{aligned}$$

or

$$P=\frac{\epsilon-1}{\epsilon}\left(\frac{\omega rB}{c}-\frac{\omega rI}{c}\right)=\frac{\epsilon-1}{\epsilon}\left(\frac{\omega rH}{c}\right)$$

and substituting we find:

$$\begin{aligned} E&=-P-P_0=-\frac{\epsilon-1}{\epsilon}\frac{\omega rH}{c}-\frac{\omega rI}{c}\\ &=-\frac{\epsilon\mu-1}{\epsilon}\frac{\omega rH}{c} \end{aligned}$$

Failure to get this expression for the field is due to the neglect of the ficitious polarization in the dielectric produced by the motion of the magnetic doublets. The factor $\frac{\mu(\epsilon-1)}{\epsilon}$ would be predicted by a theory which regards the lines of induction of the magnet as stationary. To obtain the correct result on a moving line theory requires that we take account of the translatory motion of the lines of induction attached to the individual elementary doublets.

An experiment of the kind here discussed has been performed by M. Wilson and H. A. Wilson[1] using an artificially prepared magnetic dielectric composed of steel balls imbedded in paraffin. Their results were decisively in favor of the relation given by Einstein and Laub and are in complete agreement with the present theory. It might perhaps be argued that the results of this experiment are obvious since, after all, the dielectric is made up of many small magnets in rotation about a common axis, but then it might with equal right be argued that the fallacy of the stationary line theory is equally obvious.

IV

SUMMARY

The electric field in the neighborhood of any symmetrical magnetic system spinning about its axis of symmetry is accounted for by the Maxwell-Lorentz theory to an extent determined by the correctness of certain assumptions which it is necessary to make. The field so calculated is in complete accord with all known experimental facts.

The field may, in all cases, be correctly calculated by a moving line theory in which the lines of induction of each elementary magnet of which the system is composed partake solely of the translatory part of the motion of that element.

A theory which postulates that the lines of magnetic induction rotate with the magnetic system gives incorrect results in general, but may be used to calculate the integrated value of the electromotive force around a closed conducting circuit, part of which is rotating, and part stationary.

A theory which postulates that the lines of induction stand still while the magnetic system rotates through them will yield correct results if the magnetic system is a conductor but incorrect results, in general, if it is a dielectric.

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[1] M. Wilson and H. A. Wilson, Proc. Roy. Soc. London, A, 89, 99, 1913.

PART III

EQUATIONS FOR THE DESCRIPTION OF ELECTROMAGNETIC PHENOMENA

By H. BATEMAN

I. INTRODUCTORY REMARKS ON ELECTRICITY, SPACE AND TIME

The idea of motion implies the existence of some means of recognizing again and again the entity that moves. By extending the idea of a mathematical point we have the concept of a moving point which we shall call an *electrical point* and we may start with the fundamental hypothesis that three independent quantities (α, β, γ) are sufficient to specify an electrical point and distinguish it from others. To describe its motion analytically we need a system of space and time co-ordinates. Without entering into a discussion as to the nature of these co-ordinates or the manner in which they are obtained, we may remark that the motion of the electrical points is supposed to be described in any case by equations of the form

$$x=f_1(\alpha, \beta, \gamma, t), \quad y=f_2(\alpha, \beta, \gamma, t), \quad z=f_3(\alpha, \beta, \gamma, t) \tag{1}$$

where f_1, f_2, f_3, are single-valued functions of their arguments for a definite group of electrical points. By eliminating the parameters α, β and γ we may obtain a set of differential equations

$$\frac{dx}{v_x}=\frac{dy}{v_y}=\frac{dz}{v_z}=dt \tag{2}$$

for the paths of the electrical points of a group and if the equations (1) can be solved for α, β and γ we can express the solution of these equations in the form

$$\begin{aligned} \alpha(x, y, z, t) &= \text{constant}, \\ \beta(x, y, z, t) &= \text{constant}, \\ \gamma(x, y, z, t) &= \text{constant}. \end{aligned} \tag{3}$$

When the functions α, β and γ possess continuous derivatives we may infer from (2) and (3) that

$$\rho v_x=-\frac{\partial(\alpha,\beta,\gamma)}{\partial(y,z,t)}, \quad \rho v_y=-\frac{\partial(\alpha,\beta,\gamma)}{\partial(z,x,t)}, \quad \rho v_z=-\frac{\partial(\alpha,\beta,\gamma)}{\partial(x,y,t)}, \quad \rho=\frac{\partial(\alpha,\beta,\gamma)}{\partial(x,y,z)} \tag{4}$$

where ρ is some function of x, y, z and t, which we shall call the *density* of *electricity*. This function, ρ, satisfies the *equation of continuity*

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}\left(\rho v_x\right)+\frac{\partial}{\partial y}\left(\rho v_y\right)+\frac{\partial}{\partial z}\left(\rho v_z\right)=0 \tag{5}$$

and so we shall regard electricity as a substance.

Our assumption that α, β and γ are independent implies that the quantities ρv_x, ρv_y, ρv_z and ρ are not all zero. There may also be some moving points which cannot be distinguished from each other. This case would be included in the present analysis if γ were a function of α and β. The quantities ρv_x, ρv_y, and ρv_z and ρ would then be zero and a group of such moving points could not be regarded as representing electricity; moreover, the moving points having the same values of α, β and γ would generally lie on a moving curve. We might call such points *aethereal points* and introduce two different fundamental concepts, viz., *aether* and electricity, but it seems worth while to try to avoid the introduction of an aether at the outset, and to work with only one substance, viz., electricity.

So far nothing has been said about the geometry of our groups of moving points. Are we to assume that space is Euclidean or non-Euclidean? The view is widely held that space is neither physical nor metaphysical but conventional[1] and this is the view we shall adopt in the following discussion. The conventional space, then, is Euclidean for any individual observer, and the co-ordinates used in his analytical geometry have the properties of ideal quantities which may be imagined to have been obtained by ideal measurements in an ideal medium in which the electrical points move, under normal circumstances, along straight lines with constant velocity c. An event occurs when the direction of motion of an electrical point is changed. This may be the result of a collision with some other electrical point or it may be due to some other cause at present unknown. An electrical point which is continually colliding with others may move along a zig-zag path and in the limiting case when the collisions in a finite time are infinite in number the electrical point may appear to move in a curve or straight line with a velocity less than c, or it may appear to be stationary.

We may express the matter in another way by saying that it is on account of collisions between aethereal points that electrical points exist. Let us suppose that we have two sets of ∞' aethereal points specified by the equations

$$\alpha(x, y, z, t) = \alpha_0, \qquad \beta(x, y, z, t) = \beta_0 \tag{6}$$

and

$$\beta(x, y, z, t) = \beta_0, \qquad \gamma(x, y, z, t) = \gamma_0 \tag{7}$$

respectively. Calling the points of the first set P_1, P_2, P_3, . . . and those of the second set Q_1, Q_2, Q_3, . . . we may suppose that first P collides with Q_1 at R_1; that then P_2 collides with Q_2 at R_2 and so on. Even though all the P's may be moving along straight lines with velocity c and all the Q's likewise, yet the moving point R with parameters

α_0, β_0 and γ_0 may move along a curve with a velocity less than c. This point R possesses characteristics of both the P's and the Q's and its path may be regarded as the limit of a zig-zag path which consists of first, a small portion of P_1's rectilinear path, then a portion of Q_2's rectilinear path, then a portion of P_2's path and so on. (Fig. 1.)

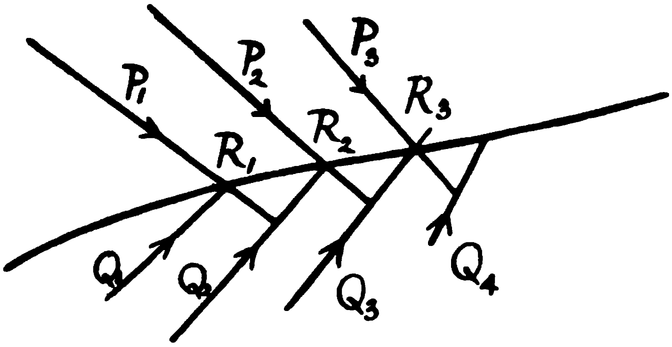

FIG. 1

We have tacitly assumed here that we really have some means of distinguishing between P_1, P_2, etc., so that these points can be regarded as electrical instead of aethereal. The parameters for P_1 may be $(\alpha_0, \beta_0, \bar{\gamma}_1)$, those for P_2 $(\alpha_0, \beta_0, \bar{\gamma}_2)$ and so on, where $\bar{\gamma}(x, y, z, t)$ is some function which is not the same as γ. In this sense the electrical point R which moves along the path R_1, R_2 . . . does not always consist of the same *fundamental electric point*, using this term to denote a point that moves with velocity c along a rectilinear path. The electrical point is in fact first P_1, then Q_2, then P_2 and so on.

The fundamental electrical points and our ideal system of space and time co-ordinates are auxiliary quantities which may be compared with parameters introduced in geometry to serve as the arguments of uniform functions occurring in the representation of quantities connected by one or more relations. Like these parameters they are probably not unique and may not have a simple physical meaning.*

We must distinguish, of course, between the ideal measurements just mentioned and actual measurements which are made with material bodies. The ideal medium and the ideal system of space-time measurements are adopted simply as the basis of a simple mathematical language and our hypothesis is that with a suitable choice of the constant, c, this language is quite adequate for the description of any conceivable phenomena. In other words, if we start with simple functions of our auxiliary parameters, we assume that we can, by suitable operations, build up more complex functions capable of representing the structure and behaviour of physical entities of an arbitrary nature. By identifying some of these complex functions with quantities that are actually measured we may eventually arrive at a theory of measurement.

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* There may not, for instance, be a correspondence between the fundamental electrical points used by one observer and those used by another.

An exact theory of measurement is difficult and so we shall first proceed on the assumption that our system of co-ordinates (x, y, z, t) is actually identical with one obtained by physical measurements and shall endeavour to formulate a scheme of equations which will account approximately for physical phenomena.

We shall assume now that x, y and z are rectangular co-ordinates and that the constant c is the velocity of light $(3\times 10^{10}\,\frac{cm.}{sec})$. We shall also regard (v_x, v_y, v_z) as the components of a velocity v which is a vector function of x, y, z, t for regions of space occupied by electricity. We shall use the vector notation of Gibbs in some of our equations and shall employ a notation that has become popular for the electric and magnetic quantities.

II. THEORY OF THE LORENTZ-FITZGERALD CONTRACTION

In attempting to formulate the equations of motion of electricity we shall be guided by the hypothesis that bodies contract when in motion in such a way that an axis in the direction of motion is $\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$ times the length of a corresponding axis of a body at rest, while axes at right angles to the direction of motion are unaltered in length.[2] This is only a rough statement of the contraction hypothesis; a more precise statement will be given later.

We shall use the symbol ρ to denote the density of the *true* electricity and v its velocity—the term true electricity being used to denote the substance made up of electrical points that travel with velocities less than c and that arise from collisions between fundamental electric points which travel along straight lines with velocity c.

We now introduce the retarded potentials

$$\Phi=\frac{1}{4\pi}\int\frac{dV}{r}[\rho]_{t-\frac{r}{c}},\quad A=\frac{1}{4\pi c}\int\frac{dV}{r}[\rho v]_{t-\frac{r}{c}},\quad \Psi=\frac{1}{4\pi}\int\frac{dV}{r}\left[\rho\sqrt{1-\frac{v^2}{c^2}}\right]_{t-\frac{r}{c}} \quad (8)$$

$$dV=d\xi d\eta d\zeta,\quad r^2=(x-\xi)^2+y-\eta)^2+(z-\zeta)^2,\quad [\rho]_{t-\frac{r}{c}}=\rho\left(\xi,\eta,\zeta,t-\frac{r}{c}\right), \quad (9)$$

the field vectors

$$H=curl\,A,\qquad E=-\frac{1}{c}\frac{\partial A}{\partial t}-\nabla\Phi \qquad (10)$$

and the tensor components

$$\begin{aligned}
W&=\frac{1}{2}(E^2+H^2)+\frac{2}{c^2}\Psi\frac{\partial^2\Psi}{\partial t^2}+\left(\frac{\partial\Psi}{\partial x}\right)^2+\left(\frac{\partial\Psi}{\partial y}\right)^2+\left(\frac{\partial\Psi}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial\Psi}{\partial t}\right)^2+C\\
S_x&=c[E_yH_z-E_zH_y]-2\Psi\frac{\partial^2\Psi}{\partial x\partial t}=c^2G_x\\
S_y&=c[E_zH_x-E_xH_z]-2\Psi\frac{\partial^2\Psi}{\partial y\partial t}=c^2G_y\\
S_z&=c[E_xH_y-E_yH_x]-2\Psi\frac{\partial^2\Psi}{\partial z\partial t}=c^2G_z\\
X_x&=\frac{1}{2}[E_x^2-E_y^2-E_z^2+H_x^2-H_y^2-H_z^2]-2\Psi\frac{\partial^2\Psi}{\partial x^2}+\Lambda\\
Y_y&=\frac{1}{2}[E_y^2-E_z^2-E_x^2+H_y^2-H_z^2-H_x^2]-2\Psi\frac{\partial^2\Psi}{\partial y^2}+\Lambda\\
Z_z&=\frac{1}{2}[E_z^2-E_x^2-E_y^2+H_z^2-H_x^2-H_y^2]-2\Psi\frac{\partial^2\Psi}{\partial z^2}+\Lambda\\
Y_z&=Z_y=E_yE_z+H_yH_z-2\Psi\frac{\partial^2\Psi}{\partial y\partial z}\\
Z_x&=X_z=E_zE_x+H_zH_x-2\Psi\frac{\partial^2\Psi}{\partial z\partial x}\\
X_y&=Y_x=E_xE_y+H_xH_y-2\Psi\frac{\partial^2\Psi}{\partial x\partial y}
\end{aligned}\tag{11}$$

$$\Lambda=\left(\frac{\partial\Psi}{\partial x}\right)^2+\left(\frac{\partial\Psi}{\partial y}\right)^2+\left(\frac{\partial\Psi}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial\Psi}{\partial t}\right)^2+C$$

The quantity W will be regarded as the density of energy, the vector S is supposed to specify the flux of energy and the vector G the momentum. The other quantities represent the six components of stress. C is a positive constant which is introduced so that W may never become negative. C has different values inside and outside electrons.

The equations of motion will be assumed to be

$$\begin{aligned}
&\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}-\frac{1}{c^2}\frac{\partial S_x}{\partial t}=0\\
&\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}-\frac{1}{c^2}\frac{\partial S_y}{\partial t}=0\\
&\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}-\frac{1}{c^2}\frac{\partial S_z}{\partial t}=0\\
&\frac{\partial S_x}{\partial x}+\frac{\partial S_y}{\partial y}+\frac{\partial S_z}{\partial z}+\frac{\partial W}{\partial t}=0
\end{aligned}\tag{12}$$

These may be written in the form

$$\begin{aligned}\rho\left[E_x+\frac{v_y}{c}H_z-\frac{v_z}{c}H_y\right]+2\Psi\frac{\partial}{\partial x}\left[\rho\sqrt{1-\frac{v^2}{c^2}}\right]&=0\\ -\rho(v\cdot E)+2\Psi\frac{\partial}{\partial t}\left[\rho\sqrt{1-\frac{v^2}{c^2}}\right]&=0\end{aligned}\tag{13}$$

and, since $\rho\left[E+\frac{1}{c}(v\times H)\right]$ is usually regarded as *the electromagnetic force per unit volume,* $2\psi\nabla\left[\rho\sqrt{-1\frac{v^2}{c^2}}\right]$ must be interpreted as *a non-electromagnetic force* which just balances the electromagnetic force.* The last equation may be regarded as *the principle of the conservation of energy*. The energy equation is thus adopted in its simplest form

$$\frac{\partial W}{\partial t}+div\ S=0$$

A more complicated form of the energy-equation has been discussed by A. Szarvassi† in relation to electromagnetic phenomena in moving media and he finds a general form of field equations with which it is consistent. His work belongs, however, to the subject of Section VIII. On account of its complexity, the energy-equation for a material medium has not been discussed in Section VIII.

If $\Psi\neq0$ the equations (13) imply that

$$\frac{d}{dt}\left[\rho\sqrt{1-\frac{v^2}{c^2}}\right]=0\tag{14}$$

where

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\tag{15}$$

This equation tells us that $\rho\sqrt{1-\frac{v^2}{c^2}}$ remains constant during the motion of the electricity. This may be regarded as a precise statement of the contraction hypothesis. The condition can evidently be satisfied if an electron has a spherical boundary and concentric spherical layers of equal density when at rest, and becomes deformed into an electron with an oblate spheroid as its boundary and concentric layers of equal density which are similar and similarly situated oblate spheroids, when set in motion, provided the axes of the spheroid are $\left(a\sqrt{1-\frac{v^2}{c^2}},\ a,\ a\right)$ when the velocity of the electron is v.

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* A previous attempt (Phys. Rev., vol. 12, 1918, p. 477) to balance the electromagnetic force by another force and reduce the equations of motion to the form (12) is unsatisfactory because the equations $Y_z = Z_y, Z_x = X_z, X_y = Y_x$ are not satisfied

† Phys. Zeitschr. Bd. 10 (1909), p. 811, Wien Ber. Ia. Bd. 119 (1910), p. 281, Bd. 120 (1911), p. 337.

When a body made up of a number of discrete electric charges is set in motion we may infer that if the body remains rigid when in uniform motion all these charges will contract in the same ratio. To conclude that the body itself contracts in the same ratio we may make use of the transformations of the theory of relativity to prove that this condition is consistent with the equations of equilibrium expressed by equations (12). It is clear that we shall also be led to the equation (14) if we multiply all the terms depending on Ψ in the tensor components by the same constant factor k. The particular choice ($k=1$) of this factor has been made so that the total radiation per period from an electric pole describing a periodic orbit may be exactly zero. We assume that this is also true, with the same value of k, for the case of an electron of finite size but we do not feel sure that this assumption is justifiable. The choice $k=\frac{1}{2}$ would lead to much simpler equations for the determination of ρ in electrostatic problems but does not lead to non-radiating orbits for the case of an electric pole.

III. THE STRUCTURE OF MATTER

Many attempts have been made to determine the form of the electron and many different hypotheses have been advanced; thus, we have the rigid electron of Abraham,[3] the deformable electron of Lorentz,[4] the ring electron of Parson[5] and Compton[6] and the electron recently invented by Page[7] in which there is a magnetic field instead of an electric field inside the electron. Attempts to account for the permanence of the electron have been made by Poincaré[8] and Page, but we shall not discuss them here; our object is to show that with a suitable law of density the electrostatic force $-\rho\nabla\Phi$ is just balanced by the new force, depending on the gradient of the density, which, in the electrostatic case, is represented by $2\Phi\nabla\rho$. It is clear that the forces balance if $\Phi=b\rho^2$ where b is a constant.[9] Assuming that the electron is bounded by a sphere of radius a and that the density, ρ, depends only on the distance r from the centre of this sphere, we assume that

$$\rho=A_0+A_2r^2+A_4r^4+\cdots \tag{16}$$

The corresponding potential is

$$\Phi=\frac{A_0}{2\cdot3}(3a^2-r^2)+\frac{A_2}{4\cdot5}(5a^4-r^4)+\frac{A_4}{6\cdot7}(7a^6-r^6)+\cdots \tag{17}$$

Putting $\Phi=b\rho^2$ and equating coefficients, we find that

$$\begin{aligned}\rho=A_0\Big[1-\frac{r^2}{12bA_0}-\frac{r^4}{720b^2A_0^2}-\frac{r^6}{10080b^3A_0^3}-\frac{31r^8}{(720)(5040)b^4A_0^4}\\-\frac{971r^{10}}{(720)^2(2310)b^5A_0^5}-\frac{53456.5\,r^{12}}{(720)^2(2940)(429)b^6A_0^6}-\cdots\Big]\end{aligned} \tag{18}$$

where

$$
\begin{aligned}
0=F\left(\frac{a^2}{bA_0}\right)\equiv 1-\frac{1}{2}\frac{a^2}{bA_0}+\frac{1}{48}\left(\frac{a^2}{bA_0}\right)^2+\frac{1}{4320}\left(\frac{a^2}{bA_0}\right)^3+\frac{1}{80640}\left(\frac{a^2}{bA_0}\right)^4 \\
+\frac{31}{(7200)(5040)}\left(\frac{a^2}{bA_0}\right)^5+\frac{971}{(720)^2(27720)}\left(\frac{a^2}{bA_0}\right)^6 \\
+\cdots \\
+\frac{53456.5}{(720)^2(2940)(6006)}\left(\frac{a^2}{bA_0}\right)^7+\cdots
\end{aligned}
\tag{19}
$$

The equation $F(x)=0$ appears to have two positive roots; the first of these is approximately 2.21 and corresponds to a law of density in which ρ diminishes slowly in absolute magnitude from a maximum value A_0 at the centre to a minimum value .8 A_0 at the boundary $r=a$. The second root, if it exists, is difficult to determine accurately by the present method† owing to the slow convergence of the series for values of x in the neighbourhood of 10. The minimum value of $|\rho|$ at the boundary is probably very much less than the maximum value of $|\rho|$ at the centre.

If the electron is supposed to be situated in an external field of constant potential Ψ_0, the equation $F\left(\frac{a^2}{bA_0}\right)=0$ must be replaced by

$$F\left(\frac{a^2}{bA_0}\right)=\frac{\Psi^0}{bA_0^2}$$

and it appears that we cannot have the same values of $\rho_{\max}$, $\rho_{\min}$ and the total charge as before. Now, on the other hand, if the electron were supposed to move from a state of equilibrium with no external field into a state of equilibruim in a field with potential Ψ_0, the quantities $\rho_{\max}$, $\rho_{\min}$, and the total charge might be expected to be the same in both cases. An explanation of this paradox may, perhaps, be obtained by considering the equilibrium of two electrons under their mutual influence. If for simplicity we imagine one electron to be the image of the other in a plane $x=0$, it is clear that equilibrium is possible only if each electron is deformed so that there are forces of attraction, arising from the density gradients, sufficient to balance the forces of repulsion.

Since ρ is constant on the boundary of the spherical electron, it must also be constant on the boundary of the deformed electron and so the boundary is an equipotential surface. The shape of the boundary may be supposed to be determined by the potential equation when the size and mutual distance of the electrons are known. The quantities to be determined are thus the size, distance apart, total charge and the constant b. On the other hand by comparing each electron with the spherical electron we obtain three equations by comparing the total charges and values of ρ_{max} and ρ_{min} but there is also a fourth equation

---

† A rough estimate, based on the first nine terms of the series, gives $x = 11$. It is thought that the smaller root may give the electron, and the larger root the proton. A more accurate value of the smaller root is 2.209012572.

$$\Phi_{max} = b(\rho_{max})^2 \tag{20}$$

which corresponds to our previous equation $F(x) = 0$. We may thus expect there to be only a finite number of solutions of the problem and so it is possible that positions of equilibrium for two electrons (or two positive charges) exist only when the linear dimensions and mutual distance are connected by certain relations.

An exact solution of the problem is much to be desired. It may be that an electron which has been in equilibrium when alone can only be in equilibrium in certain particular fields and that some readjustment of charge is necessary before it can be in equilibrium in an arbitrary field. It seems likely, too, that a state of steady motion of a system of discrete charges may be possible only when the size of each orbit is related in some way to the size of each charge. The present analysis may, then, lead eventually to Bohr's theory of the atom, but it is full of difficulties.

If b and A_0 are given constant negative values and a negative constant potential Ψ_0 is added to (17) it appears that equilibrium is possible only with a smaller value of a and a smaller total charge. This does not mean, however, that an electron would necessarily lose charge when brought into a field with a negative potential. The effect of the negative potential would be, in fact, to cause a shrinkage. This shrinkage could not simply continue until a new state of equilibrium, with a different value of b, was reached because $\rho\sqrt{1-v^2/c^2}$ remains constant If $|\rho|$ increases, v must increase also. Unfortunately we cannot follow these steps which occur when an electron passes from one field to another and so we cannot say at present whether the electron loses charge or not. If two electrons were made to approach one another we might expect both to shrink and be deformed; but it is necessary to take into account the nature of the agency which causes the approach. If it is simply speed of translation, we may expect the kinetic energy to be transformed into the potential energy associated with the increased masses of the two electrons.

A point of some interest is, that the binding forces which hold together a number of electric charges of the same sign can apparently be weakened by decreasing the absolute magnitude of the potential Ψ. It may be that a radioactive transformation occurs when a large number of nuclear (or external) electrons accidently concentrate in the immediate neighbourhood of one portion of the nucleus[10] and weaken the binding forces by lowering the positive potential. It is unlikely that radioactivity can be controlled[*] because a large negative potential is probably needed to bring the positive potential down to a critical value. It is hardly likely that a sufficiently high negative potential would be obtained when a number of electrons in a substance approached one another as a result of the heat motion.

At a high temperature when many electrons have been driven out of a substance, the positive potential in the neighbourhood of an atomic nucleus may be larger than usual and if the pressure is also very great we may have the right conditions for the formation of complex nuclei. The complex elements may thus be formed in the stars.

The present analysis thus raises many interesting questions that need answering and may eventually indicate the reason why all electrons carry approximately the same charge. It also suggests the possibility that in a chemical compound we may have two electrons close together and in equilibrium under their mutual attractions and repulsions, both electrons being deformed. Such a pair of electrons might act as a chemical bond between two atoms. According to Lewis[11] and Langmuir[12] the pair of electrons is one of the most stable groups.

It should be mentioned that experiments on the deflection of α-particles by atomic nuclei give the usual inverse square law of repulsion, because the α-particles travel at such a high speed that there is not enough time for the α-particle and atomic nucleus to become deformed when they are close together and give rise to mutual attractions sufficient to appreciably modify the repulsion. If an atomic nucleus and an α-particle could be made to approach each other slowly there might be some chance of a combination.

IV. THE FIELD OF A MOVING ELECTRIC POLE

Exact expressions for the electromagnetic potentials belonging to the field of a moving electric pole were deduced from the corresponding potentials for a continuous distribution of electricity by A. Liénard[13] and E. Wiechert,[14] approximate forms having been used previously by J. Larmor.[15] The exact potentials are

$$A=e\frac{v}{M}, \quad \Phi=e\frac{c}{M} \tag{21}$$

where v is the velocity of the pole at time τ, $[\xi(\tau), \eta(\tau)\zeta(\tau)]$ are the rectangular co-ordinates of the pole at this instant, $[\xi'(\tau), \eta'(\tau), \zeta'(\tau)]$ its component velocities, $-4\pi e$ its electric charge, and M a quantity defined by the equations

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[*] We are thinking here of control by means of agents that do not owe their existence to radioactivity. Rutherford has succeeded in breaking up some atoms with the aid of α-particles.

$$\begin{aligned} M &= [x-\xi(\tau)]\xi'(\tau)+[y-\eta(\tau)]\eta'(\tau)+[z-\zeta(\tau)]\zeta'(\tau)-c^2(t-\tau), \\ &= r(v_r-c), \end{aligned} \tag{22}$$

$$r^2=[x-\xi(\tau)]^2+[y-\eta(\tau)]^2+[z-\zeta(\tau)]^2=c^2(t-\tau)^2. \tag{23}$$

The last equation, combined with the inequality $\tau<t$, associates a time τ with each space-time point (x, y, z, t). If the velocity v is always less than c the time τ associated with (x, y, z, t) is unique and increases as t increases if (x, y, z) remains stationary or moves with a velocity less than c. The point $[\xi(\tau), \eta(\tau), \zeta(\tau)]$ is called the *effective position*[16] of the pole for (x, y, z, t). The scalar potential Ψ is now

$$\Psi=e\frac{\sqrt{c^2-v^2}}{M} \tag{24}$$

and is connected with the potentials A and Φ by the relation[17]

$$\Psi=\sqrt{\Phi^2-A^2} \tag{25}$$

The potentials Ψ, A and Φ, moreover, satisfy the equations

$$\Box\,\Psi=0,\quad div\,A+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0,\quad \Box A=0,\quad \Box\Phi=0. \tag{26}$$

where

$$\Box U=\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}+\frac{\partial^2U}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2U}{\partial t^2} \tag{27}$$

Defining the field vectors E and H by means of the equations

$$H=curl\,A,\quad E=-\frac{1}{c}\frac{\partial A}{\partial t}-\nabla\Phi \tag{28}$$

it is found that they satisfy Maxwell's equations

$$\begin{aligned} curl\,H&=\frac{1}{c}\frac{\partial E}{\partial t}, & div\,E&=0 \\ curl\,E&=-\frac{1}{c}\frac{\partial H}{\partial t}, & div\,H&=0 \end{aligned} \tag{29}$$

and that

$$\Box\,E=0,\quad \Box\,H=0. \tag{30}$$

We also have the relation

$$e^2(E^2-H^2)=\Psi^4 \tag{31}$$

and we may write[18]

$$eE = \Psi^2 \frac{s_1 - s_2}{1-(s_1 \cdot s_2)}, \quad eH = \Psi^2 \frac{s_2 \times s_1}{1-(s_1 \cdot s_2)} \tag{32}$$

where s_1 and s_2 are the two real unit vectors which satisfy the relations

$$\begin{aligned} E+(s\times H) &= s(s\ .\ E), \quad H-(s\times E) = s(s\ .\ H) \\ (s\times E).(s\times H) &= 0, \qquad\qquad (s\times E)^2 = (s\times H)^2 \end{aligned} \tag{33}$$

The vector s_1 is in the direction of the radius from the effective position of the electric pole. The vector s_2 does not generally admit of a simple geometrical interpretation, but when the pole moves with uniform velocity along a straight line, s_2 is in the direction of the radius to that position of the pole which could be hit by a bullet travelling from (x, y, z, t) along a straight line with velocity c.

The field vectors may also be expressed in Hargreaves' form[19]

$$H = \nabla\tau \times \nabla\sigma, \quad E = \frac{1}{c}\left[\frac{\partial\sigma}{\partial t}\nabla\tau - \frac{\partial\tau}{\partial t}\nabla\sigma\right] \tag{34}$$

where

$$\sigma = e\frac{K}{M} \tag{35}$$

and

$$K = (x-\xi)\xi'' + (y-\eta)\eta'' + (z-\zeta)\zeta'' + c^2 - \xi'^2 - \eta'^2 - \zeta'^2. \tag{36}$$

These expressions are useful for many purposes. It may be remarked in passing that

$$\frac{1}{c^2}\frac{\partial\sigma}{\partial t}\frac{\partial\tau}{\partial t} - \frac{\partial\sigma}{\partial x}\frac{\partial\tau}{\partial x} - \frac{\partial\sigma}{\partial y}\frac{\partial\tau}{\partial y} - \frac{\partial\sigma}{\partial z}\frac{\partial\tau}{\partial z} = \sqrt{E^2 - H^2} = -(s \cdot E) = \frac{\Psi^2}{e} \tag{37}$$

The field vectors may also be expressed in the form

$$E = \nabla\alpha \times \nabla\beta, \quad H = \frac{1}{c}\left[\frac{\partial\alpha}{\partial t}\nabla\beta - \frac{\partial\beta}{\partial t}\nabla\alpha\right] \tag{38}$$

where α and β are certain functions of x, y, z and t. The equations $\alpha = \text{const}$, $\beta = \text{const}$ may be regarded as those of a moving line of electric force.[20] In the case of an electric pole moving in an arbitrary manner along the axis of x, we may write[21]

$$\alpha = c\frac{x-\xi-v(t-\tau)}{c^2(t-\tau)-v(x-\xi)}, \quad \beta = \tan^{-1}\frac{y}{z} \tag{39}$$

The lines of electric force have been found in a number of other cases—for instance, in uniform circular motion, in uniform helical motion, in Born's hyperbolic motion and in motions that can be derived from these by means of transformations of certain types.

A line of electric force may be regarded as the locus of a series of points moving along straight lines with velocity c and projected from the moving pole at successive instants, the direction of projection varying according to a law which has been formulated by Leigh Page[22] and the author. If the unit vector s specifies the direction of projection at time τ, the components of s satisfy differential equations which are embodied in the vector equation

$$(c^2-v^2)\frac{ds}{d\tau}=(v-cs)(s\cdot v')+v'[c-(s\cdot v)] \tag{40}$$

This equation may be replaced by a Riccation equation[23] with the single complex dependent variable

$$w=\frac{s_x+i\,s_y}{1+s_z} \tag{41}$$

Many interesting geometrical properties of the lines of force and the associated directions of projection have been given by Leigh Page,[22] F. D. Murnaghan[24] and the author.[25]

The rate of radiation of energy from a moving electric pole may be found by an extension of the method given by Liénard[13], use being now made of the more general tensor components defined in Section II. It is found that the rate of radiation is

$$\frac{4\pi e^2c}{3}\frac{d^2}{d\tau^2}\left(\frac{1}{c^2-v^2}\right) \tag{42}$$

the contribution of the electromagnetic radiation, as found by Liénard and Larmor[15] being

$$\frac{8\pi e^2c}{3}\left[\frac{v'^2}{(c^2-v^2)^2}+\frac{(v\,v')^2}{(c^2-v^2)^3}\right] \tag{43}$$

the remaining part arising from the radiation which depends on the variation of the function Ψ.

In a periodic motion the total radiation of energy per period is zero—a result which is in accordance with the idea of non-radiating orbits that has been used so successfully in atomic theory. The total radiation may also be zero in a non-periodic orbit, if we integrate from apse to apse, an apse being defined as a point on the path where the velocity is a maximum or minimum.

V. THE REFLECTION OF LIGHT AT A MOVING PLANE MIRROR

Let the equation of the moving mirror be $x=ut$ and let us consider in the first place the effect of the mirror on the field of a moving electric pole whose co-ordinates at time τ are (ξ, η, ζ).

We shall assume that the effect is the same as if an additional field were produced by an electric pole at the image of the moving pole, the image of (ξ, η, ζ, τ) being supposed to be determined by the equations[26]

$$\left.\begin{aligned} \xi^* &= \xi - \frac{2c^2}{c^2-u^2}(\xi - u\tau) \qquad & \eta^* &= \eta \\ \tau^* &= \tau - \frac{2u}{c^2-u^2}(\xi - u\tau) \qquad & \zeta^* &= \zeta \end{aligned}\right\} \tag{44}$$

which give

$$\xi^* - u\tau^* = -(\xi - u\tau) \tag{45}$$

and so furnish the ordinary laws of reflexion for points that move with the same velocity as the mirror. If (x^*, y^*, z^*, t^*) are derived from (x, y, z, t) by the same set of equations, we have

$$\left.\begin{aligned} &(x^*-\xi^*)^2+(y^*-\eta^*)^2+(z^*-\zeta^*)^2-c^2(t^*-\tau^*)^2 \\ &=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2-c^2(t-\tau)^2 \end{aligned}\right\} \tag{46}$$

Placing a charge e at (ξ, η, ζ, τ) and a charge $-e$ at $(\xi^*, \eta^*, \zeta^*, \tau^*)$, it is easily verified that, with the notation of Section IV

$$d(\tau^*, \sigma^*) = -d(\tau, \sigma) \tag{47}$$

where the symbol $d(\tau, \sigma)$ is used for an expression of type $d\tau\delta\sigma - d\sigma\delta\tau$, the d's and δ's referring to independent increments.

The above equation may be written in the form

$$\left.\begin{aligned} &H_x^* d(y^*, z^*) + H_y^* d(x^*, x^*) + H_z^*(x^*, y^*) + cE_x^* d(x^*, t^*) + cE_y^* d(y^*, t^*) \\ &\quad + cE_z^* d(z^*, t^*) + H_x d(y, z) + H_y d(z, x) + H_z d(x, y) + cE_x d(x, t) \\ &\quad + cE_y d(y, t) + cE_z d(z, t) = 0. \end{aligned}\right\} \tag{48}$$

At points of the moving mirror, however, we have $x^* = x$, $y^* = y$, $z^* = z$, $t^* = t$, and the above equation becomes

$$\left.\begin{aligned} &\overline{H}_x d(y, z) + \overline{H}_y d(z, x) + \overline{H}_z d(x, y) + c\overline{E}_x d(x, t) + c\overline{E}_y d(y, t) \\ &\quad + c\overline{E}_z d(z, t) = 0 \end{aligned}\right\} \tag{49}$$

where $\overline{E}$ and $\overline{H}$ are the electric and magnetic forces in the total field. This is the condition to be satisfied at the surface of a perfect conductor or perfect reflector. We also have $\overline{\Psi} = 0$. These conditions hold for arbitrary fields, as may be seen by superposition of the fields of a number of moving electric poles.

We may deduce from equation (48) that

$$\left.\begin{aligned} E_x^* &= E_x & H_x^* &= -H_x \\ E_y^* &= -\frac{c^2+u^2}{c^2-u^2}E_y + \frac{2cu}{c^2-u^2}H_z, & H_y^* &= \frac{c^2+u^2}{c^2-u^2}H_y + \frac{2cu}{c^2-u^2}E_z, \\ E_z^* &= -\frac{c^2+u^2}{c^2-u^2}E_z - \frac{2cu}{c^2-u^2}H_y, & H_z^* &= \frac{c^2+u^2}{c^2-u^2}H_z - \frac{2cu}{c^2-u^2}E_y. \end{aligned}\right\} \tag{50}$$

$$\left.\begin{aligned} E_y^*-\frac{u}{c}H_z^* &= -\left(E_y-\frac{u}{c}H_z\right)\\ E_z^*+\frac{u}{c}H_y^* &= -\left(E_z+\frac{u}{c}H_y\right)\\ H_y^*+\frac{u}{c}E_z^* &= H_y+\frac{u}{c}E_z\\ H_z^*+\frac{u}{c}E_y^* &= H_z-\frac{u}{c}E_y \end{aligned}\right\} \tag{51}$$

These equations give us the usual boundary conditions at the surface of a perfect conductor, viz.,

$$\overline{H}_x=0,\ \overline{E}_y=\frac{u}{c}\overline{H}_z,\ E_z+\frac{u}{c}\overline{H}_y=0.$$

These equations and the equation $\Psi=0$ indicate that in the total field there is no flow of energy across the surface of the mirror.

We also have

$$\begin{aligned} &E_x^*d(y^*,z^*)+E_y^*d(z^*,x^*)+E_z^*d(x^*,y^*)-cH_x^*d(x^*,t^*)-cH_y^*(y^*,t^*)\\ &\quad -cH_z^*d(z^*,t^*)=E_xd(y,z)+E_yd(z,x)+E_zd(x,y)-cH_xd(x,t)\\ &\quad -cH_yd(y,t)-cH_zd(z,t)\ (51) \end{aligned}$$

or

$$d(\alpha^*,\beta^*)=d(\alpha,\beta). \tag{52}$$

This equation may be interpreted to mean that the lines of electric force of the pole $(\xi^*, \eta^*, \zeta^*, \tau^*)$ are the images in the moving mirror of the lines of electric force of the moving pole (ξ, η, ζ, τ). It is important to notice that if the point (ξ, η, ζ, τ) moves with a velocity less than c, the point $(\xi^*, \eta^*, \zeta^*, \tau^*)$ does also. The image in the mirror of a stationary observer is an observer moving with velocity

$$v=\frac{2c^2u}{c^2+u^2} \tag{53}$$

in the direction of the axis of x. Referred to the stationary axes this observer will suffer from the Lorentz-Fitzgerald contraction

$$\sqrt{1-\frac{v^2}{c^2}}:1.$$

as is easily seen from the equations.

All observations made by the moving observer may be regarded as images in the mirror of corresponding observation's made by the stationary observer and we may deduce the whole theory of the relativity transformation from reflections in moving mirrors.

It is important to notice that a mirror and an associated set of moving points and images reflect into a mirror and an associated set of moving points and images provided all mirrors move uniformly. This result is of importance in the theory of the Michelson and Morley experiment, for the essential feature of the arrangement of mirrors is that *A* is the image of B in *C*.

This implies that successive reflections in *C* and *A* are equivalent to successive reflections in *B* and *C*. Since *A* is the image of *B* in *C* whatever be the direction of the Earth's uniform motion,* the explanation of the negative result of the experiment is, perhaps, clear, for any instantaneous event which happens at a stationary or moving point *P*,which we suppose to be a source of light, has the same instantaneous image whether the light is reflected first in *C* and then in *A* or first in *B* and then in *C*. This is true for different positions of the apparatus relative to the Earth's direction of motion and so the conditions for interference are the same in all cases.

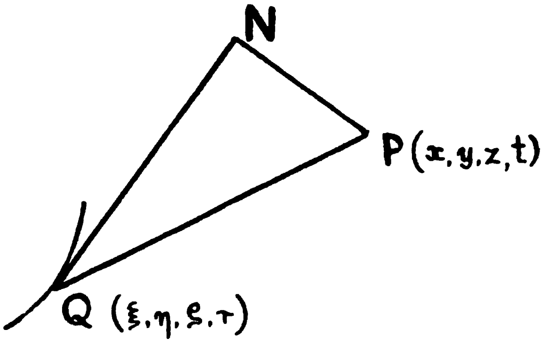

FIGURE 2.

The requirement that successive reflections in *C* and *A* are equivalent to successive reflections in *B* and *C* is satisfied when the mirrors are in uniform rectilinear motion relative to the standard axes of coordinates for which the velocity of light is *c* provided *A* is the image of *B* in *C* when measurements are made by an observer who is at rest relative to the mirrors. It is not necessary for the mirrors *A* and *B* to be perpendicular; it is simply sufficient that *C* should bisect the angle between the planes *A* and *B*, *when the angles are measured by an observer moving with the same velocity as the mirrors.* The source of light *P* can have any motion whatever.

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[*] A. Righi, Comptes Rendus, t. 168 (1919), p. 837, t. 170, pp. 497, 1550, concludes that a nul result is to be expected for any orientation of the apparatus relative to the earth's direction of translational motion.

VI. Behaviour of Electrical Quantities Under a Relativity Transformation

In order that a clear idea may be obtained of the mathematical properties of the quantities that have been introduced we shall indicate the behaviour of these quantities under the group of Lorentzian transformations,[27] a typical transformation of the group being

$$x^1=\frac{x-wt}{\sqrt{1-\frac{w^2}{c^2}}},\quad t^1=\frac{t-\frac{wx}{c^2}}{\sqrt{1-\frac{w^2}{c^2}}},\quad y^1=y,\quad z^1=z. \tag{54}$$

The quantities $\Psi, \rho\sqrt{1-\frac{v^2}{c^2}}, E^2-H^2, (E\cdot H)$ *and* Φ^2-A^2 are invariants. So also are

$$\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2$$

$$\rho[c\Phi-(A.v)]$$

$$(x-\xi)^2+(y-y)^2+(z-\zeta)^2-c^2(t-\tau)^2$$

$$dx^2+dy^2+dz^2-c^2dt^2.$$

$$d(\alpha,\beta),\quad d(\sigma,\tau),\quad d(x,y,z,t)$$

$$A_xdx+A_ydy+A_zdz-c\Phi dt$$

$$H_xd(y,z)+H_yd(z,x)+H_zd(x,y)+cE_xd(x,t)+cE_yd(y,t)+cE_zd(z,t)$$

$$E_xd(y,z)+E_yd(z,x)+E_zd(x,y)-cH_xd(x,t)-cH_yd(y,t)-cH_zd(z,t)$$

The vectors v, cs_1 and cs_2 transform like velocities. If we use the Poincaré-Minkowski representation of a space-time point (x, y, z, t) by a point with rectangular co-ordinates (x, y, z, ict) in a space of four dimensions S_4, a moving point is represented by a curve or *world line* in S_4 and we may introduce the direction cosines (w_1, w_2, w_3, w_4) of the tangent at a point of this curve. These direction cosines are connected with the velocity of the moving point at the corresponding instant by the equations.[28]

$$iw_1=\frac{v_x}{(c^2-v^2)^{\frac{1}{2}}},\quad iw_2=\frac{v_y}{(c^2-v^2)^{\frac{1}{2}}},\quad iw_3=\frac{v_z}{(c^2-v^2)^{\frac{1}{2}}},\quad iw_4=\frac{ic}{(c^2-v^2)^{\frac{1}{2}}} \tag{55}$$

and of course

$$w_1^2+w_1^2+w_3^2+w_4^2=1 \tag{56}$$

With the aid of these direction cosines Minkowski[21] was able to express the potentials of a moving point charge in the symmetrical form

$$A_x=\frac{ew_1}{R_w},\quad A_y=\frac{ew_2}{R_w},\quad A_z=\frac{ew_3}{R_w},\quad i\Phi=\frac{ew_4}{R_w} \tag{57}$$

where

$$R_w=w_1(x-\xi)+w_2(y-\eta)+w_3(z-\zeta)+icw_4(t-\tau) \tag{58}$$

is the projection of the radius QP on the tangent QN to the world line.

We also have

$$i\Psi = \frac{e}{R_w} \qquad (59)$$

The function Ψ is in fact analogous to a function used by M. Abraham[21] in his theory of gravitation.

It was pointed out by E. B. Wilson[21] and G. N. Lewis and also later by Megh Nad Saha[22] that the potentials can be written in the form

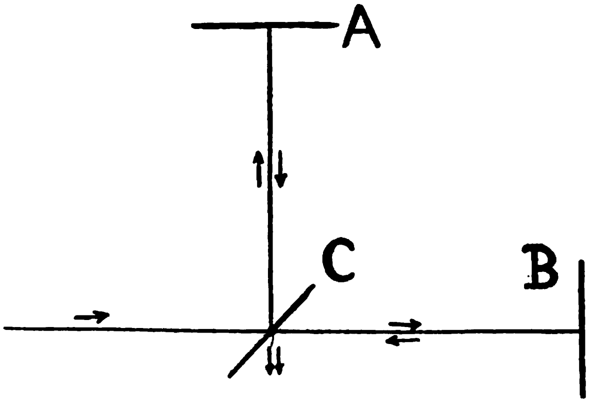

FIGURE 3.

$$A_x = \frac{iew_1}{P}, \quad A_y = \frac{iew_2}{P}, \quad A_z = \frac{iew_3}{P}, \quad i\Phi = \frac{iew_4}{P} \qquad (60)$$

where P is the perpendicular distance of P from the tangent at Q, so that

$$P^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 - c^2(t-\tau)^2 - [w_1(x-\xi) + w_2(y-\eta) + w_3(z-\zeta) + icw_4(t-\tau)]^2$$

It is clear that $P = iR_w$ on account of the relation (23).

The field vectors E and H may also be given a geometrical interpretation. To see this let Q, R and S be three consecutive points on the world line of an electric pole, and let Q be the point in S_4 which corresponds to the effective position of the pole for a space-time point (x, y, z, t) represented in S_4 by P. The line PQ is a minimal or isotropic line, i. e. a line of zero length.

Consider the sphere which passes through $PQRS$. It will have a tangent plane at P and in this tangent plane there will be two isotropic lines, viz PQ and a line PT. This line PT has a direction corresponding to a velocity cs_2 while QP has a direction corresponding to the velocity cs_1. The unit vectors s_1 and s_2 and the scalar Ψ needed for the expressions (32) for E and H thus admit of a geometrical interpretation.

The six components of E and iH can be regarded as $\frac{\Psi^2}{e}$ times the direction cosines of the tangent plane at P to the sphere just mentioned.

In the case of uniform rectilinear motion the sphere reduces to a plane and it can be shown that the field vectors at (x, y, z, t) can be expressed in terms of the positional co-ordinates of the electric pole at any time. if these are (ξ, η, ζ) at time τ we may write[22]

$$E_x - i\, H_x = -e(c^2\tau^2 - \xi^2 - \eta^2 - \zeta^2)\frac{X}{R^3} \tag{62}$$

where

$$X = c(x\tau - \xi t) + i(y\zeta - z\eta), \qquad R^2 = X^2 + Y^2 + Z^2 \tag{63}$$

VII. FIELDS WITH SINGULARITIES OF A COMPLEX NATURE WHICH MOVE WITH VELOCITIES LESS THAN C.

The general type of electromagnetic field with an isolated simple singularity or pole which moves with a velocity less than c may be specified by means of the equations

$$H + iE = Q = curl\ C - \frac{i}{c}\frac{\partial C}{\partial t} - i\nabla\Gamma \tag{64}$$

$$C = -\frac{m}{4\pi}\frac{v}{M}, \qquad \Gamma = -\frac{m}{4\pi}\frac{c}{M} \tag{65}$$

where $m = h + ie$ is a complex constant, $iC = B + iA$, $i\Gamma = \Omega + i\Phi$. The potentials B and Ω are the potentials of magnetic type while A and Φ are the potentials of electric type. Equating the real and imaginary parts in equation (64) we have

$$H = curl\ A - \nabla\Omega - \frac{1}{c}\frac{\partial B}{\partial t}$$

$$E = -curl\ B - \nabla\Phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

We may call e the electric charge and h the magnetic charge associated with the moving pole. When $h = o$ we obtain the formulae of Liénard.

It should be noticed that

$$div\ C + \frac{1}{c}\frac{\partial \Gamma}{\partial t} = 0, \qquad div\ A + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0, \qquad div\ B + \frac{1}{c}\frac{\partial \Omega}{\partial t} = 0 \tag{66}$$

We may also write

$$\Pi = -\frac{m}{4\pi}\frac{\sqrt{c^2 - v^2}}{M}, \qquad i\Pi = \Lambda + i\Psi \tag{67}$$

and so introduce a function Λ which is the magnetic analogue of Ψ.

The mathematical analysis for fields with complex singularities may be developed in a simple manner by using contour integrals of type

$$\frac{f(\tau)}{M} = \frac{i}{\pi}\int_c \frac{f(s)ds}{[x - \xi(s)]^2 + [y - \eta(s)]^2 + [z - \zeta(s)]^2 - c^2(t - s)^2} \tag{68}$$

as recommended by A. W. Conway,[34] G. Herlotz[35] and others. The proof that our potentials C, Γ, Π are wave-functions satisfying $\Box C = 0$, $\Box\Gamma = 0$, $\Box\Pi = 0$, $div\ C + \frac{1}{c}\frac{\partial\Gamma}{\partial t} = 0$ is immediate and it can also be shown at once that if f is an arbitrary function of τ, we have the identity

$$\frac{f'(\tau)}{M}=\frac{\partial}{\partial x}\left(\frac{f\xi'}{M}\right)+\frac{\partial}{\partial y}\left(\frac{f\eta'}{M}\right)+\frac{\partial}{\partial z}\left(\frac{f\zeta'}{M}\right)+\frac{\partial}{\partial t}\left(\frac{f}{M}\right) \tag{69}$$

To obtain fields with isolated moving singularities of higher order we consider a neighboring pole whose position at time τ is given by the equations

$$\begin{aligned}\xi_1(\tau)&=\xi(\tau)+\theta\alpha(\tau)+\frac{1}{2}\theta^2\lambda(\tau)+\frac{1}{6}\theta^3\Phi(\tau)+\\ \eta_1(\tau)&=\eta(\tau)+\theta\beta(\tau)+\frac{1}{2}\theta^2\mu(\tau)+\frac{1}{6}\theta^3\psi(\tau)+\\ \zeta_1(\tau)&=\zeta(\tau)+\theta\gamma(\tau)+\frac{1}{2}\theta^2\nu(\tau)+\frac{1}{6}\theta^3\chi(\tau)+\end{aligned} \tag{70}$$

Forming potentials C_1, Γ_1, Π_1 in the same way as before and expressing them as contour integrals of Conway's type, we find on expanding them in ascending powers of the small constant θ and writing $C^*=C-C_1$, $\Gamma^*=\Gamma-\Gamma_1$, $\Pi^*=\Pi-\Pi_1$, that

$$\begin{aligned}\frac{4\pi}{m}C_x{}^*=&\theta\left[\frac{\alpha'}{M}-\frac{\partial}{\partial x}\left(\frac{\alpha\xi'}{M}\right)-\frac{\partial}{\partial y}\left(\frac{\beta\xi'}{M}\right)-\frac{\partial}{\partial z}\left(\frac{\gamma\xi'}{M}\right)\right]\\ &+\frac{1}{2}\theta^2\left[\frac{\lambda'}{M}-2\frac{\partial}{\partial x}\left(\frac{\alpha'\alpha}{M}\right)-2\frac{\partial}{\partial y}\left(\frac{\alpha'\beta}{M}\right)-2\frac{\partial}{\partial z}\left(\frac{\alpha'\gamma}{M}\right)-\frac{\partial}{\partial x}\left(\frac{\lambda\xi'}{M}\right)-\frac{\partial}{\partial y}\left(\frac{\mu\xi'}{M}\right)\right.\\ &-\frac{\partial}{\partial z}\left(\frac{\nu\xi'}{M}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\alpha^2\xi^1}{M}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\beta^2\xi'}{M}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{\gamma^2\xi'}{M}\right)+2\frac{\partial^2}{\partial y\partial z}\left(\frac{\beta\gamma\xi'}{M}\right)\\ &\left.+2\frac{\partial^2}{\partial z\partial x}\left(\frac{\gamma\alpha\xi'}{M}\right)+2\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha\beta\xi'}{M}\right)\right]+\frac{1}{6}\theta^3\left[\frac{\phi'}{M}-3\frac{6}{\partial x}\left(\frac{\lambda'\alpha}{M}\right)\right.\\ &-3\frac{\partial}{\partial y}\left(\frac{\gamma'\beta}{M}\right)-3\frac{\partial}{\partial z}\left(\frac{\lambda'\gamma}{M}\right)-3\frac{\partial}{\partial x}\left(\frac{\alpha'\lambda}{M}\right)-3\frac{\partial}{\partial y}\left(\frac{\alpha'\mu}{M}\right)-3\frac{\partial}{\partial z}\left(\frac{\alpha'\nu}{M}\right)\\ &-\frac{\partial}{\partial x}\left(\frac{\phi\xi'}{M}\right)-\frac{\partial}{\partial y}\left(\frac{\psi\xi'}{M}\right)-\frac{\partial}{\partial z}\left(\frac{\chi\xi'}{M}\right)+3\frac{\partial^2}{\partial x^2}\left(\frac{\alpha'\alpha^2}{M}\right)+3\frac{\partial^2}{\partial y^2}\left(\frac{\alpha'\beta^2}{M}\right)\\ &+3\frac{\partial^2}{\partial z^2}\left(\frac{\alpha'\gamma^2}{M}\right)+6\frac{\partial^2}{\partial y\partial z}\left(\frac{\alpha'\beta\gamma}{M}\right)+6\frac{\partial^2}{\partial z\partial x}\left(\frac{\alpha'\gamma\alpha}{M}\right)+6\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha'\alpha\beta}{M}\right)\\ &+3\frac{\partial^2}{\partial x^2}\left(\frac{\xi'\alpha\lambda}{M}\right)+3\frac{\partial^2}{\partial y^2}\left(\frac{\xi'\beta\mu}{M}\right)+3\frac{\partial^2}{\partial z^2}\left(\frac{\xi'\gamma\nu}{M}\right)+3\frac{\partial^2}{\partial y\partial z}\left[\frac{\xi'(\beta\nu+\gamma\mu)}{M}\right]\\ &+3\frac{\partial^2}{\partial z\partial x}\left\{\frac{\xi'(\gamma\lambda+\alpha\nu)}{M}\right\}+3\frac{\partial^2}{\partial x\partial y}\left\{\frac{\xi'(\alpha\mu+\beta\lambda)}{M}\right\}-\frac{\partial^3}{\partial x^3}\left\{\frac{\xi'\alpha^3}{M}\right\}\\ &-\frac{\partial^3}{\partial y^3}\left\{\frac{\xi'\beta^3}{M}\right\}-\frac{\partial^3}{\partial z^3}\left\{\frac{\xi'\gamma^3}{M}\right\}-3\frac{\partial^3}{\partial y^2\partial z}\left\{\frac{\xi'\beta^2\gamma}{M}\right\}-3\frac{\partial^3}{\partial y\partial z^2}\left\{\frac{\xi'\beta\gamma^2}{M}\right\}\\ &-3\frac{\partial^3}{\partial z^2\partial x}\left\{\frac{\xi'\gamma^2\alpha}{M}\right\}-3\frac{\partial^3}{\partial z\partial x^2}\left\{\frac{\xi'\alpha^2\gamma}{M}\right\}-3\frac{\partial'}{\partial x^2\partial y}\left\{\frac{\xi'\alpha^2\beta}{M}\right\}\\ &-3\frac{\partial^3}{\partial x\partial y^2}\left\{\frac{\xi'\alpha\beta^2}{M}\right\}-6\frac{\partial^3}{\partial x\partial y\partial z}\left\{\frac{\xi'\alpha\beta\gamma}{M}\right\}\end{aligned} \tag{71}$$

with corresponding expressions for Γ^* and Π^*. Making use of the identity (69) we find that we may write

$$\frac{4\pi}{m}C^*=\frac{\partial L}{\partial t}+curl\ N, \tag{72}$$

where

$$\begin{aligned}L_z=\theta\frac{\alpha}{M}+\frac{1}{2}\theta^2\Big[\frac{\lambda}{M}-\frac{\partial}{\partial x}\Big(\frac{\alpha^2}{M}\Big)-\frac{\partial}{\partial y}\Big(\frac{\beta\alpha}{M}\Big)-\frac{\partial}{\partial z}\Big(\frac{\gamma\alpha}{M}\Big)\Big]+\frac{1}{6}\theta^3\Big[\frac{\phi}{M}\\
-3\frac{\partial}{\partial x}\Big(\frac{\alpha\lambda}{M}\Big)-3\frac{\partial}{\partial y}\Big(\frac{\beta\lambda}{M}\Big)-3\frac{\partial}{\partial z}\Big(\frac{\gamma\lambda}{M}\Big)+\frac{\partial^2}{\partial x^2}\Big(\frac{\alpha^3}{M}\Big)+\frac{\partial^2}{\partial y^2}\Big(\frac{\alpha\beta^2}{M}\Big)\\
+\frac{\partial^2}{\partial z^2}\Big(\frac{\alpha\gamma^2}{M}\Big)+2\frac{\partial^2}{\partial y\partial z}\Big(\frac{\alpha\beta\gamma}{M}\Big)+2\frac{\partial^2}{\partial z\partial x}\Big(\frac{\alpha^2\gamma}{M}\Big)+2\frac{\partial^2}{\partial x\partial y}\Big(\frac{\alpha^2\beta}{M}\Big)\Big]\end{aligned} \tag{73}$$

$$\begin{aligned}N_z=\theta\frac{\beta\zeta'-\gamma\eta'}{M}+\frac{1}{2}\theta^2\Big[\frac{\mu\zeta'-\nu\eta'}{M}+\frac{\beta\gamma'-\beta'\gamma}{M}+\frac{\partial}{\partial x}\Big\{\frac{\alpha(\eta'\gamma-\zeta'\beta)}{M}\Big\}\\
+\frac{\partial}{\partial y}\Big\{\frac{\beta(\eta'\gamma-\zeta'\beta)}{M}\Big\}+\frac{\partial}{\partial z}\Big\{\frac{\gamma(\eta'\gamma-\zeta'\beta)}{M}\Big\}\Big]+\frac{1}{6}\theta^3\Big[\frac{\psi\zeta'-\chi\eta'}{M}\\
+3\frac{\mu\gamma'-\nu\beta'}{M}+3\frac{\partial}{\partial x}\Big\{\frac{\alpha(\eta'\nu-\zeta'\mu)}{M}\Big\}+3\frac{\partial}{\partial y}\Big\{\frac{\beta(\eta'\nu-\zeta'\mu)}{M}\Big\}\\
+3\frac{\partial}{\partial z}\Big\{\frac{\gamma(\eta'\nu-\zeta'\mu)}{M}\Big\}+2\frac{\partial}{\partial x}\Big\{\frac{\alpha(\beta'\gamma-\beta\gamma')}{M}\Big\}+2\frac{\partial}{\partial y}\Big\{\frac{\beta(\beta'\gamma-\beta\gamma')}{M}\Big\}\\
+2\frac{\partial}{\partial z}\Big\{\frac{\gamma(\beta'\gamma-\beta\gamma')}{M}\Big\}+\frac{\partial^2}{\partial x^2}\Big\{\frac{\alpha^2(\beta\zeta'-\gamma\eta')}{M}\Big\}+\frac{\partial^2}{\partial y^2}\Big\{\frac{\beta^2(\beta\zeta'-\gamma\eta')}{M}\Big\}\\
+\frac{\partial^2}{\partial z^2}\Big\{\frac{\gamma^2(\beta\zeta'-\gamma\eta')}{M}\Big\}+2\frac{\partial^2}{\partial y\partial z}\Big\{\frac{\beta\gamma(\beta\zeta'-\gamma\eta')}{M}\Big\}+2\frac{\partial^2}{\partial z\partial x}\Big\{\frac{\gamma\alpha(\beta\zeta'-\gamma\eta')}{M}\Big\}\\
+2\frac{\partial^2}{\partial x\partial y}\Big\{\frac{\alpha\beta(\beta\zeta'-\gamma\eta')}{M}\Big\}\Big]+\end{aligned} \tag{74}$$

We may also write*

$$\frac{4\pi}{m}\Gamma^*=-c\ div\ L \tag{75}$$

$$\begin{aligned}\Pi^*=\theta\Big[\frac{\partial}{\partial x}\Big(\frac{\alpha\kappa}{M}\Big)+\frac{\partial}{\partial y}\Big(\frac{\beta\kappa}{M}\Big)+\frac{\partial}{\partial z}\Big(\frac{\gamma\kappa}{M}\Big)+\frac{\xi'\alpha'+\eta'\beta'+\zeta'\gamma'}{\kappa M}\Big]+\frac{1}{2}\theta^2\Big[\frac{\partial}{\partial x}\Big(\frac{\lambda\kappa}{M}\Big)\\
+\frac{\partial}{\partial y}\Big(\frac{\mu\kappa}{M}\Big)+\frac{\partial}{\partial z}\Big(\frac{\nu\kappa}{M}\Big)-2\frac{\partial}{\partial x}\Big(\frac{\alpha\sigma}{\kappa M}\Big)-2\frac{\partial}{\partial y}\Big(\frac{\beta\sigma}{\kappa M}\Big)-2\frac{\partial}{\partial z}\Big(\frac{\gamma\sigma}{\kappa M}\Big)-\frac{\partial^2}{\partial x^2}\Big(\frac{\alpha^2\kappa}{M}\Big)\\
-\frac{\partial^2}{\partial y^2}\Big(\frac{\beta^2\kappa}{M}\Big)-\frac{\partial^2}{\partial z^2}\Big(\frac{\gamma^2\kappa}{M}\Big)-2\frac{\partial^2}{\partial y\partial z}\Big(\frac{\beta\gamma\kappa}{M}\Big)-2\frac{\partial^2}{\partial z\partial x}\Big(\frac{\gamma\alpha\kappa}{M}\Big)-2\frac{\partial^2}{\partial x\partial y}\Big(\frac{\alpha\beta\kappa}{M}\Big)\\
+\frac{\omega}{\kappa M}+\frac{\sigma^2}{\kappa^3 M}\Big]+\frac{1}{6}\theta^3\Big[\frac{\partial^3}{\partial x^3}\Big(\frac{\alpha^3\kappa}{M}\Big)+\frac{\partial^3}{\partial y^3}\Big(\frac{\beta^3\kappa}{M}\Big)+\frac{\partial^3}{\partial z^3}\Big(\frac{\gamma^3\kappa}{M}\Big)+3\frac{\partial^3}{\partial y^2\partial z}\Big(\frac{\beta^2\gamma\kappa}{M}\Big)\end{aligned}$$

---

* These cumbersome expressions are needed for a calculation of the radiation from the number of moving electric poles. The results of the calculation will be published later.

$$
\begin{aligned}
&+3\frac{\partial^3}{\partial y\partial z^2}\left(\frac{\beta\gamma^2\kappa}{M}\right)+3\frac{\partial^3}{\partial z^2\partial x}\left(\frac{\gamma^2\alpha\kappa}{M}\right)+3\frac{\partial^3}{\partial z\partial x^2}\left(\frac{\gamma\alpha^2\kappa}{M}\right)+3\frac{\partial^3}{\partial x^2\partial y}\left(\frac{\alpha^2\beta\kappa}{M}\right)\\
&+3\frac{\partial^3}{\partial x\partial y^2}\left(\frac{\alpha\beta^2\kappa}{M}\right)+6\frac{\partial^3}{\partial x\partial y\partial z}\left(\frac{\gamma\alpha\beta\kappa}{M}\right)-3\frac{\partial^2}{\partial x^2}\left(\frac{\alpha\lambda\kappa}{M}\right)-3\frac{\partial^2}{\partial y^2}\left(\frac{\beta\mu\kappa}{M}\right)\\
&-3\frac{\partial^2}{\partial z^2}\left(\frac{\gamma\nu\kappa}{M}\right)-3\frac{\partial^2}{\partial y\partial z}\left\{\frac{(\beta\nu+\gamma\mu)\kappa}{M}\right\}-3\frac{\partial^2}{\partial z\partial x}\left\{\frac{(\gamma\lambda+\alpha\nu)\kappa}{M}\right\}\\
&-3\frac{\partial^2}{\partial x\partial y}\left\{\frac{(\alpha\mu+\beta\lambda)\kappa}{M}\right\}+3\frac{\partial^2}{\partial x^2}\left(\frac{\alpha^2\sigma}{\kappa M}\right)+3\frac{\partial^2}{\partial y^2}\left(\frac{\beta^2\sigma}{\kappa M}\right)+3\frac{\partial^2}{\partial z^2}\left(\frac{\gamma^2\sigma}{\kappa M}\right)\\
&+6\frac{\partial^2}{\partial y\partial z}\left(\frac{\beta\gamma\sigma}{\kappa M}\right)+6\frac{\partial^2}{\partial z\partial x}\left(\frac{\gamma\alpha\sigma}{\kappa M}\right)+6\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha\beta\sigma}{\kappa M}\right)+\frac{\partial}{\partial x}\left(\frac{\phi\kappa}{M}\right)+\frac{\partial}{\partial y}\left(\frac{\psi\kappa}{M}\right)\\
&+\frac{\partial}{\partial z}\left(\frac{\chi\kappa}{M}\right)-3\frac{\partial}{\partial x}\left(\frac{\lambda\sigma}{\kappa M}\right)-3\frac{\partial}{\partial y}\left(\frac{\mu\sigma}{\kappa M}\right)-3\frac{\partial}{\partial z}\left(\frac{\nu\sigma}{\kappa M}\right)-3\frac{\partial}{\partial x}\left(\frac{\omega\alpha}{\kappa M}\right)-3\frac{\partial}{\partial y}\left(\frac{\omega\beta}{\kappa M}\right)\\
&-3\frac{\partial}{\partial z}\left(\frac{\omega\gamma}{\kappa M}\right)-3\frac{\partial}{\partial x}\left(\frac{\alpha\sigma^2}{\kappa^3 M}\right)-3\frac{\partial}{\partial y}\left(\frac{\beta\sigma^2}{\kappa^3 M}\right)-3\frac{\partial}{\partial z}\left(\frac{\gamma\sigma^2}{\kappa^3 M}\right)+3\frac{\sigma^2}{\kappa^5 M}+\frac{3\sigma\omega}{\kappa^3 M}\\
&\left.+\frac{3(\alpha'\lambda'+\beta'\mu'+\gamma'\nu')+\xi'\phi'+\eta'\psi'+\zeta'\chi'}{\kappa M}\right]+\cdots\cdot \qquad (76)
\end{aligned}
$$

where

$$
\left.\begin{aligned}
\sigma&=\xi'\alpha'+\eta'\beta'+\zeta'\gamma',\kappa=\sqrt{c^2-v^2}\\
\omega&=\alpha'^2+\beta'^2+\gamma'^2+\xi'\lambda'+\eta'\mu'+\zeta'\nu'.
\end{aligned}\right. \qquad (77)
$$

Since L and N satisfy $\Box L=0$. $\Box N=0$, it is easy to verify that we have the identities

$$curl\ C^*=-\frac{i}{c}\frac{\partial D}{\partial t}-i\nabla\Sigma \qquad (78)$$

$$\frac{1}{c}\frac{\partial C^*}{\partial t}+\nabla\Gamma^*=i\ curl\ D \qquad (79)$$

where

$$
\left.\begin{aligned}
D&=\frac{m}{4\pi}\left[ic\ curl\ \mathrm{L}-\frac{i}{c}\frac{\partial N}{\partial t}\right]\\
\Sigma&=\frac{im}{4\pi}\ div\ N
\end{aligned}\right\} \qquad (80)
$$

Hence, if we write

$$
\left.\begin{aligned}
C^*+D=C_0,\quad \Gamma^*+\Sigma=\Gamma_0\\
\frac{m}{4\pi}(-cL+iN)=G_0
\end{aligned}\right\} \qquad (81)
$$

we may specify an electromagnetic field by means of the equations

$$Q_0=H_0+iE_0=curl\ C_0=-\frac{i}{c}\left[\frac{\partial C_0}{\partial t}+c\nabla\Gamma_0\right] \qquad (82)$$

$$C_0=\frac{1}{c}\frac{\partial G_0}{\partial t}+i\ curl\ G_0,\quad \Gamma_0=-div\ G_0 \qquad (83)$$

where G_0 is a generalized Hertzian vector which satisfies the wave-equation $\Box G_0=0$. The field thus represented is really the difference of the fields of the two neighboring poles.

We must now examine in detail the fields obtained by separating the different powers of θ. Let us write

$$G_0=\theta\, G_1+\frac{1}{2}\theta^2 G_2+\frac{1}{6}\theta^3 G_3+\cdot\cdot\cdot \tag{84}$$

and use a to denote the vector with components (α, β, γ), then we may regard the field derived from G_1, as the electromagnetic field of the combination of a moving electric doublet of electric moment ae and a moving magnetic doublet of magnetic moment ah. The combination may be called an *electromagnetic doublet*, or dipole.

The field derived from G_2 consists of two parts, one of which is the field of an electromagnetic doublet, whose complex moment $g_1=q_1+ip_1$ is given by

$$g_1+\frac{i}{c}[v\times g_1]=m\left\{b+\frac{i}{c}[v\times b]+\frac{i}{c}[a'\times a]\right\} \tag{85}$$

where b denotes the vector with components (λ, μ, ν) and a' the vector with components $(\alpha', \beta', \gamma')$. The other part of the field may be regarded as the field of a combination of quadrupoles whose physical properties are described by means of a diad with components

$$\begin{matrix} \alpha g_x & \beta g_x & \gamma g_x \\ \alpha g_y & \beta g_y & \gamma g_y \\ \alpha g_z & \beta g_z & \gamma g_z \end{matrix}$$

g being the complex moment ma.

The field derived from G_3 consists of three parts which may be regarded as the fields of an electromagnetic dipole, a combination of quadrupoles and a combination of octupoles. The moment of the dipole is g_2 where

$$g_2+\frac{i}{c}[v\times g_2]=m\left\{k+\frac{i}{c}[a'\times b]\right\} \tag{86}$$

k being the vector with component (ϕ, ψ, χ). The diad specifying the properties of the combination of quadrupoles has the components

$$\begin{matrix} \alpha(m\lambda+g_{1x}) & \beta(m\lambda+g_{1x}) & \gamma(m\lambda+g_{1x}) \\ \alpha(m\mu+g_{1y}) & \beta(m\mu+g_{1y}) & \gamma(m\mu+g_{1y}) \\ \alpha(m\nu+g_{1z}) & \beta(m\nu+g_{1z}) & \gamma(m\nu+g_{1z}) \end{matrix}$$

The physical properties of the combination of octupoles may be specified by means of a triad with 27 components which may be represented by the scheme

$$g_x\begin{bmatrix} \alpha^2\ \alpha\beta\ \alpha\gamma \\ \alpha\beta\ \beta^2\ \beta\gamma \\ \alpha\gamma\ \beta\gamma\ \gamma^2 \end{bmatrix},\ g_y\begin{bmatrix} \alpha^3,\ \alpha\beta,\ \alpha\gamma \\ \alpha\beta,\ \beta^2,\ \beta\gamma \\ \alpha\gamma,\ \beta\gamma,\ \gamma^2 \end{bmatrix},\ g_z\begin{bmatrix} \alpha^2\ \alpha\beta\ \alpha\gamma \\ \alpha\beta\ \beta^2\ \beta\gamma \\ \alpha\gamma\ \beta\gamma\ \gamma^2 \end{bmatrix}$$

It should be noticed that on account of the velocity of the singularity (ξ, η, ζ) an electromagnetic moment g_s enters into the expressions for the potentials of different orders in the form of a derived vector

$$f_s = g_s + \frac{i}{c}[v \times g_s] \tag{87}$$

and the effective diads and triads are obtained by replacing the g's in the above expressions by f's. If p and q are electric and magnetic moments respectively, we can say that the *effective moments* are

$$p + \frac{i}{c}[v \times q] \text{ and } q - \frac{i}{c}[v \times p]$$

respectively.

It should be noticed that in the case of the field derived from G_2 there is a magnetic moment even when $h = 0$, i. e., when there are no magnetic poles.

According to the electron theory the origin of magnetism is accounted for in this way* but according to the recent theory of Ewing[36] it is necessary to consider electrical systems equivalent to magnetic quadrupoles to account for the magnetic properties of iron. Such systems cannot be specified in this point singularity method by means of Hertzian vectors of types G_1 and G_2; it is necessary to use Hertzian vectors of higher order and so the rather cumbersome expression for G_2 is not without interest. It may be remarked that our definition of the magnetic moment differs slightly from the one usually adopted, but the present definition seems the most natural one as it preserves the usual symmetry; it also gives the usual definition when $v = 0$.

When $\xi = \eta = \zeta = 0$ for all values of τ so that the primary singularity is stationary, our results take a simpler form. We now have $M = -cr$, where r is the distance from the origin. All quantities such as $\alpha, \beta, \gamma, \lambda, \mu, \nu, \theta, \phi, \psi$ and their derivatives are functions of $t - \frac{r}{c}$. We thus find that

$$\begin{aligned} -cL_x = {} & \theta\frac{\alpha}{r} + \frac{1}{2}\theta^2\left[\frac{\lambda}{r} - \frac{\partial}{\partial x}\left(\frac{\alpha^2}{r}\right) - \frac{\partial}{\partial y}\left(\frac{\beta\alpha}{r}\right) - \frac{\partial}{\partial z}\left(\frac{\gamma\alpha}{r}\right)\right] \\ & + \frac{1}{6}\theta^3\left[\frac{\phi}{r} - 3\frac{\partial}{\partial x}\left(\frac{\alpha\lambda}{r}\right) - 3\frac{\partial}{\partial y}\left(\frac{\beta\lambda}{r}\right) - 3\frac{\partial}{\partial z}\left(\frac{\gamma\lambda}{r}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\alpha^3}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\alpha\beta^2}{r}\right)\right. \\ & \left. + \frac{\partial^2}{\partial z^2}\left(\frac{\alpha\gamma^2}{r}\right) + 2\frac{\partial^2}{\partial y \partial z}\left(\frac{\alpha\beta\gamma}{r}\right) + 2\frac{\partial^2}{\partial z \partial x}\left(\frac{\alpha^2\gamma}{r}\right) + 2\frac{\partial^2}{\partial x \partial y}\left(\frac{\alpha^2\beta}{r}\right)\right] + \cdots \end{aligned} \tag{88}$$

$$\begin{aligned} -cN_x = {} & \frac{1}{2}\theta^2\left[\frac{\beta\gamma' - \beta'\gamma}{r}\right] + \frac{1}{6}\theta^3\left[3\frac{\mu\gamma' - \nu\beta'}{r} + 2\frac{\partial}{\partial x}\left\{\frac{\alpha(\beta'\gamma - \beta\gamma')}{r}\right\}\right. \\ & \left. + 2\frac{\partial}{\partial y}\left\{\frac{\beta(\beta'\gamma - \beta\gamma')}{r}\right\} + 2\frac{\partial}{\partial z}\left\{\frac{\gamma(\beta'\gamma - \beta\gamma')}{r}\right\}\right] \end{aligned} \tag{89}$$

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* See, for instance, H. A. Lorentz, "Theory of Electrons," and "Vorträge über kinetische Theorie der Materie," Leipzig, Teubner (1914); H. J. van Leeuwen, Diss. Leiden, (1919).

$$
\begin{aligned}
-\Pi = \theta\left[\frac{\partial}{\partial x}\left(\frac{\alpha}{r}\right)+\frac{\partial}{\partial y}\left(\frac{\beta}{r}\right)+\frac{\partial}{\partial z}\left(\frac{\gamma}{r}\right)\right]+\frac{1}{2}\theta^2\left[\frac{\partial}{\partial x}\left(\frac{\lambda}{r}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{r}\right)+\frac{\partial}{\partial z}\left(\frac{\nu}{r}\right)\right.\\
-\frac{\partial^2}{\partial x^2}\left(\frac{\alpha^2}{r}\right)-\frac{\partial^2}{\partial y^2}\left(\frac{\beta^2}{r}\right)-\frac{\partial^2}{\partial z^2}\left(\frac{\gamma^2}{r}\right)-2\frac{\partial^2}{\partial y\partial z}\left(\frac{\beta\gamma}{r}\right)-2\frac{\partial^2}{\partial z\partial x}\left(\frac{\gamma\alpha}{r}\right)-2\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha\beta}{r}\right)\\
\left.+\frac{\alpha'^2+\beta'^2+\gamma'^2}{c^2r}\right]+\frac{1}{6}\theta^3\left[\frac{\partial^3}{\partial x^3}\left(\frac{\alpha^3}{r}\right)+\frac{\partial^3}{\partial y^3}\left(\frac{\beta^3}{r}\right)+\frac{\partial^3}{\partial z^3}\left(\frac{\gamma^3}{r}\right)+3\frac{\partial^3}{\partial y^2\partial z}\left(\frac{\beta^2\gamma}{r}\right)\right.\\
+3\frac{\partial^3}{\partial y\,\partial z^2}\left(\frac{\beta\gamma^2}{r}\right)+3\frac{\partial^3}{\partial z^2\,\partial x}\left(\frac{\gamma^2\alpha}{r}\right)+3\frac{\partial^3}{\partial z\partial x^2}\left(\frac{\gamma\alpha^2}{r}\right)+3\frac{\partial^3}{\partial x^2\partial y}\left(\frac{\alpha^2\beta}{r}\right)\\
+3\frac{\partial^3}{\partial x\partial y^2}\left(\frac{\alpha\beta^2}{r}\right)+6\frac{\partial^3}{\partial x\partial y\partial z}\left(\frac{\alpha\beta\gamma}{r}\right)-3\frac{\partial^2}{\partial x^2}\left(\frac{\alpha\lambda}{r}\right)-3\frac{\partial^2}{\partial y^2}\left(\frac{\beta\mu}{r}\right)-3\frac{\partial^2}{\partial z^2}\left(\frac{\gamma\nu}{r}\right)\\
-3\frac{\partial^2}{\partial y\partial z}\left\{\frac{(\beta\nu+\gamma\mu)}{r}\right\}-3\frac{\partial^2}{\partial z\partial x}\left\{\frac{(\gamma\lambda+\alpha\nu)}{r}\right\}-3\frac{\partial^2}{\partial x\partial y}\left\{\frac{(\alpha\mu+\beta\lambda)}{r}\right\}\\
+\frac{\partial}{\partial x}\left(\frac{\phi}{r}\right)+\frac{\partial}{\partial y}\left(\frac{\psi}{r}\right)+\frac{\partial}{\partial z}\left(\frac{\chi}{r}\right)-3\frac{\partial}{\partial x}\left(\frac{\omega\alpha}{c^2r}\right)-3\frac{\partial}{\partial y}\left(\frac{\omega\beta}{c^2r}\right)-3\frac{\partial}{\partial z}\left(\frac{\omega\gamma}{c^2r}\right)\\
\left.+3\frac{\alpha'\lambda'+\beta'\mu'+\gamma'\nu'}{c^2r}\right]+\cdots
\end{aligned}
\tag{90}
$$

where $\omega=\alpha'^2+\beta'^2+\gamma'^2$.

Let us now calculate the rate of radiation of energy in an electromagnetic field defined by the equations

$$
\left.
\begin{aligned}
&Q=H+iE=curl\ C=-\frac{i}{c}\left[\frac{\partial C}{\partial t}+c\nabla\Gamma\right]\\
&C=\frac{i}{c}\frac{\partial G}{\partial t}+i\ curl\ G,\quad \Gamma=-div\ G\\
&G=\frac{g}{r}+\sum_1^3\frac{\partial}{\partial x_n}\left(\frac{g_n}{r}\right)+\sum_1^3\sum_1^3\frac{\partial^2}{\partial x_m\partial x_n}\left(\frac{g_{mn}}{r}\right)+\cdots
\end{aligned}
\right\}
\tag{91}
$$

where the vectors g, g_n, g_{mn}, etc., are functions of $t-\frac{r}{c}$ and $x_1=x, x_2=y_1$, $x_3=z$. In calculating $H+iE$ we need only terms of order $\frac{1}{r}$, consequently we need only differentiate the vector g.

Let s be a unit vector in the direction of the radius r from the origin, then $\frac{\partial}{\partial x}g_n=-\frac{1}{c}g'_n s_x$. When we differentiate again we need not differentiate $s_x=\frac{x}{r}$ because this would introduce a term of order $\frac{1}{r}$ and make the product of order $\frac{1}{r^2}$. We may therefore treat the components of s as constants in the subsequent differentiations.

Writing $s_x=s_1$, $s_y=s_2$, $s_z=s_3$, we may use the expression

$$
G=\frac{1}{r}\left[g-\frac{1}{c}\Sigma s_n g'_n+\frac{1}{c^2}\Sigma\Sigma s_m s_n g''_{mn}\right]
\tag{92}
$$

and we find that, here*

$$H+iE=-\frac{1}{c^2r}\Big[s\times g''-\frac{1}{c}(s\times\Sigma s_n g_n''')+\frac{1}{c^2}(s\times\Sigma\Sigma s_{mn}\, g_{mn}'''')$$

$$-i\Big\{s\times(s\times g'')-\frac{1}{c}s\times(s\times\Sigma s_n g_n''')+\frac{1}{c^2}s(\times s\times\Sigma\Sigma s_{mn} g_{mn}'''')\Big\}\Big] \qquad (93)$$

Now if $H+iE=s\times a$ and $H-iE=s\times\bar{a}$, we have

$$2i(E\times H)=s[s\cdot(a\times\bar{a})]=s[\bar{a}\cdot(s\times\bar{a})] \qquad (94)$$

Now if $a=b-i(s\times b)$ where $(s.b)=0$, we have

$$(s\times a)=(s\times b)+ib=ia \qquad (95)$$

Hence in this case the radiation in the direction of s vanishes only if $(a.\bar{a})-0$, i. e., if a is a vector *zero* of magnitude; the condition $(s.b)=0$ is satisfied if

$$-c^2rb=g''-s(s\cdot g'')-\frac{1}{c}\Big\{\Sigma s_n g_n'''-s\Big(s\cdot\Sigma s_n g_n'''\Big)\Big\}+\cdots \qquad (96)$$

and we have

$$H+iE=s\times a, \quad a=b-i(s\times b).$$

Hence, if the radiation in the direction of s vanishes, b must be a vector which satisfies the relation

$$b=i(s\times b) \qquad (97)$$

i. e., b must be a vector in the direction of one of the isotropic lines at right angles to s. This implies that

$$s\times g''-\frac{1}{c}\Big\{s\times\Sigma s_n g_n'''\Big\{+\frac{1}{c^2}\Big\{s\times\Sigma\Sigma s_{mn} g_{mn}''''\Big\}-\cdots$$

is a vector in the direction of one of the isotropic lines just mentioned We may therefore write

$$g''-\frac{1}{c}\Sigma s_n g_n'''+\frac{1}{c^2}\Sigma\Sigma s_{mn} g_{mn}''''-\cdots$$
$$=ks+u \qquad (98)$$

where u is an isotropic vector. This implies that G is of the form

$$G=curl\, S+\frac{i}{c}\frac{\partial S}{\partial t}+\nabla T \qquad (99)$$

---

* An expression for the field strength, E, at a great distance from the origin has been published by Leigh Page, Phys. Rev., Vol. 19, (1922), p. 423.

where

$$\left.\begin{aligned} S&=\frac{w}{r}+\sum_{n}\frac{w_n}{r}+\sum_{m}\sum_{n}\frac{w_{mn}}{r}+\\ T&=\frac{u}{r}+\sum_{n}\frac{u_n}{r}+\sum_{m}\sum_{m}\frac{u_{mn}}{r}+ \end{aligned}\right\} \tag{100}$$

the vectors w, w_n, w_{mn}, . . . and the scalars u, u_n, u_{mn} . . . being fuctions of $t-\frac{r}{c}$. It is clear that $\Box\, S=0$, $\Box T=0$.

When G has the above form, however, we find that $E\equiv 0$, $H\equiv 0$ and the field vanishes altogether.

The radiation can also vanish when $g''=0$, $g_n'''=0$, etc., but these equations imply that the electrical system consists of charges moving uniformly along straight lines and in this case the electromagnetic radiation is known to vanish. The electromagnetic radiation is known to vanish also in the case of a complete ring of electric charges in steady motion and in certain other cases [37] but these are of no interest for the purposes of the quantum theory.

Our conclusion is that the most satisfactory way of getting rid of the radiation of an electron describing an atomic orbit is to introduce the forces and radiation depending on the scalar potential Ψ.

In the case of a simple electric doublet whose moment is always parallel to OZ, we have

$$A_x=A_y=0,\quad A_z=\frac{1}{c}\frac{\partial}{\partial t}\frac{g_z}{r},\quad \Phi=\Psi=-div\frac{g}{r}=-\frac{\partial}{\partial z}\frac{g_z}{r} \tag{101}$$

Retaining only the important terms we have

$$\left.\begin{aligned} E_x&=\frac{1}{c^2r}g_z''\frac{xz}{r^2}, & H_x&=-\frac{y}{c^2r^2}g_z'' & 2\Psi\frac{\partial^2\Psi}{\partial x\partial t}&=-\frac{2xz^2}{c^3r^5}g_z'g_z'''\\ E_y&=\frac{1}{c^2r}g_y''\frac{yz}{r^2}, & H_y&=\frac{x}{c'r'}g_z'' & 2\Psi\frac{\partial^2\Psi}{\partial y\partial t}&=-\frac{2yz^2}{c^3r^5}g_z'g_z'''\\ E_z&=-\frac{1}{c^2r}g_y''\frac{x^2+y^2}{r^2}, & H_z&=0 & 2\Psi\frac{\partial^2\Psi}{\partial z\partial t}&=-\frac{2z^3}{c^3r^5}g_z'g_z''' \end{aligned}\right\} \tag{102}$$

The total rate of radiation is thus

$$\frac{1}{c^3}\int_0^{\pi}\int_0^{2\pi}Sin\ \theta d\theta d\phi\left[(g_z'')^2\ Sin^2\theta+2g_z'g_z'''\ Cos^2\ \theta\right]$$

$$=\frac{8\pi}{3c^3}\left[g_z''^2+g_z'g_z'''\right]=\frac{8\pi}{3c^3}\frac{d}{dt}\left[g_z'g_z''\right] \tag{103}$$

The energy radiated in a period is thus evanescent. This result may be extended to the case in which 3 components of the moment vary.

VIII. Field Equations for a Moving Dielectric

We may pass from a medium containing isolated point charges, dipoles, quadrupoles, etc., to a medium containing a continuous volume density of electric charges, dipoles, quadrupoles, etc., by replacing the point potentials of Section VII by potentials of the type used in Section II.

Assuming that there are no magnetic poles, we write

$$C=\frac{1}{c}\frac{\partial G}{\partial t}+i\,curl\,G, \quad \Gamma=-div\,G \tag{104}$$

$$H+iE=curl\,C+curl\,A-\frac{i}{c}\frac{\partial A}{\partial t}-i\nabla\Phi \tag{105}$$

where A and Φ are defined as in Section II and G is of the form

$$\begin{aligned}G=\frac{1}{4\pi}\int\frac{dV}{r}[f]_{t-\frac{r}{c}}+\frac{1}{4\pi}\sum_{m=1}^{3}\frac{\partial}{\partial x_m}\int\frac{dV}{r}[f_m]_{t-\frac{r}{c}}\\+\frac{1}{4\pi}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2}{\partial x_m\partial x_n}\int\frac{dV}{r}[f_{mn}]_{t-\frac{r}{c}}+\ \cdots\end{aligned} \tag{106}$$

where (x_1, x_2, x_3) are written for (x, y, z) respectively and the complex vectors f, f_m, f_{mn}, etc., have the forms

$$\begin{aligned}f&=g+\frac{i}{c}(v\times g) & g&=q+ip\\ f_m&=g_m+\frac{i}{c}(v\times g_m) & g_m&=q_m+ip_m\\ f_{mn}&=g_{mn}+\frac{i}{c}(v\times g_{mn}) & g_{mn}&=q_{mn}+ip_{mn}\end{aligned} \tag{107}$$

Since

$$\Box\int\frac{dV}{r}[f]_{t-\frac{r}{c}}=-4\pi f \tag{108}$$

we find on substituting these expressions in (105) that

$$\left.\begin{aligned}curl\,H&=\frac{1}{c}\left[\frac{\partial D}{\partial t}+\rho v\right], & div\,D&=\rho\\ curl\,E&=-\frac{1}{c}\frac{\partial B}{\partial t}, & div\,B&=0\end{aligned}\right\} \tag{109}$$

$$\begin{aligned}B=H+q-\frac{1}{c}(v\times p)+\sum_{m=1}^{3}\frac{\partial}{\partial x_m}\left[q_m-\frac{1}{c}(v\times p_m)\right]+\\ \sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2}{\partial x_m\partial x_n}\left[q_{mn}-\frac{1}{c}(v\times p_{mn})\right]+\ \cdots\end{aligned} \tag{110}$$

$$D=E+p+\frac{1}{c}(v\times q)+\sum_{m=1}^{3}\frac{\partial}{\partial x_m}\left[p_m+\frac{1}{c}(v\times q_m)\right]+$$
$$\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2}{\partial x_m\partial x_n}\left[p_{mn}+\frac{1}{c}(v\times q_{mn})\right]+\ \dots \qquad (111)$$

It is thought that with suitable conventions as to the meaning of the quantities involved, these equations can be regarded as the electromagnetic equations for a material medium in which there are no conduction currents.

In writing the equations in the above form our plan has been to eliminate as far as possible all terms which might be interpreted as volume densities of electric current. Thus when B and E are treated as the fundamental vectors, the term in curl H which depends on $-\frac{1}{c}$ (v$\times p$) might be described as a Röntgen current, while according to our convention $-\frac{1}{c}(v\times p)$ is a part of the *effective magnetisation* per unit volume. This convention is similar to that recommended by Minkowski and Born.[38]

It should be noticed that Heaviside units have been used in these equations. The transition to electrostatic units is easily made. Thus the first equation would have the factor $\frac{4\pi}{c}$ instead of $\frac{1}{c}$, in the third equation all the p's and q's would be multiplied by the 4π and in the last equation E would be divided by 4π.

H. A. Lorentz obtained a set of electromagnetic equations for the description of electromagnetic phenomena in a material substance by a process of averaging,[39] in which the quantities averaged were the field vectors occurring in the fundamental equations of the theory of electrons,

$$curl\ d=-\frac{1}{c}\frac{\partial h}{\partial t},\qquad div\ h=0 \qquad (112)$$

$$curl\ h=\frac{1}{c}\left[\frac{\partial d}{\partial t}+\rho v\right],\qquad div\ d=\rho \qquad (113)$$

which, in their turn, can be regarded as derived from the ether equations

$$\left.\begin{aligned} curl\ d_0&=-\frac{1}{c}\frac{\partial h_0}{\partial t}, & div\ h_0&=0\\ curl\ h_0&=\frac{1}{c}\frac{\partial d_0}{\partial t}, & div\ d_0&=0\end{aligned}\right\} \qquad (114)$$

by a process of averaging in which electric poles are considered.

Lorentz's plan was to obtain an average value of a quantity for a point P by averaging the quantity over a sphere S whose center is at P and whose radius is small in the physical sense and yet so large in comparison with the distance between electrons that S can contain an enormous number of electrons. Writing

$$\frac{1}{S}\int d\cdot dS=E, \quad \frac{1}{S}\int h\cdot dS=B \tag{115}$$

the equations (112) give

$$curl\ E=-\frac{1}{c}\frac{\partial B}{\partial t}, \quad div\ B=0 \tag{116}$$

but equations (113) take a modified form depending upon the form in which we express the average values of ρ and ρv.

This process of averaging has been developed and extended by other writers in various ways and has been combined with the kinematics of the theory of relativity so as to give equations for electromagnetic phenomena in moving bodies that are compatible with that theory. It has been remarked, however, by Fokker[40] that the introduction of the ideas of relativity is unnecessary and simply complicates the analysis. The desired equations can be obtained directly by a variational process invented by Minkowski[8] and developed by Born,[38] and the invariance or the equations in the transformations of the theory of relativity can be inferred from their final form. The theory in which electrons are treated simply as point charges in the process of averaging has been worked out in a simple manner by Cunningham,[41] using both space and time averages separately, together with the kinematics of relativity. The use of an average over a space-time domain is a feature of the work of Dällenbach[42] and Fokker.[40] In some respects this method of averaging has advantages over the others, especially when one is dealing with moving matter.

The results that may be obtained by the methods of averaging appear to be no different in character from those which may be derived by the methods of Section VII and Section VIII. Fokker's variational equation (3) corresponds to our equation (69), in fact in Section VII we have really used the variational method in a slightly different form. Our analysis differs from that of Fokker and others because, for the sake of generality, we have introduced magnetic poles and doublets and have examined the detailed structure of the terms corresponding to singularities of higher order. The introduction of magnetic doublets in the terms derived from G_1, which correspond to Fokker's first variational terms, seems desirable in view of the recent ideas with regard to the magnetic properties of the electron and the experimental evidence in favour of the magneton, i. e., an electron with a magnetic moment. In the usual theory magnetization arises from the revolution of electrons round the positive nucleus of an atom and its effects appear in the second order variation, i. e., in the terms corresponding to those which arise from our function G_2. If, however, the electron itself has a magnetic moment the magnetic terms should appear at an earlier stage as in the case of terms derivable from G_1, and should differ in structure from the terms appearing in the ordinary theory. This difference in structure would, perhaps, be most easily detected in the case of crystals and some work has already been done in this direction with results that seem favourable from the point of view of the magneton theory.

The introduction of singularities of order higher than that of a dipole may seem unnecessary for the representation of the ordinary properties of a dielectric but it may be mentioned that W. Voigt[43] has used a volume distribution of quadrupoles to represent the piezoelectric properties of a crystal having a center of symmetry. The piezoelectric properties of a crystal without a center of symmetry are thought by him to be represented with sufficient accuracy by a volume distribution of ordinary dipoles. In dealing with the piezoelectric and magnetic properties of crystals it seems best[*] to make use of the dynamical theory of crystal lattices that has been developed by Born[44] and other writers. This theory seems also to be the best method of studying the propagation of waves of light in an isotropic or doubly refracting medium especially when attention has to be paid to the phenomena of dispersion and rotary polarisation.

IX. CONSTITUTIVE RELATIONS

The introduction of empirical constitutive relations into the theory of electromagnetism commenced with the work of Poisson[45] and may be regarded as an application, or rather an extension, of Hooke's law of the linear dependence of stress upon strain when the latter is small. In Poisson's theory of magnetism, the magnetic induction B and the magnetic force H are connected by the simple relation $B = \mu H$, where μ is the permeability of the magnetic substance. It is now known that this theory is inadequate when there are permanent magnets and strong fields,[46] for in the latter case we have the phenomenon of hysteresis.[*] In Weber's theory of magnetism it is supposed that a substance contains minute particles, each of which acts as a magnet, and that in the proces of magnetising a ferromagnetic substance these are turned into more or less complete alignment. Ewing[36] has recently added to this the idea that the thing that turns is neither a molecule nor an atom but something within the atom, whose orientation is to some extent controlled by pairs of fixed magnets, pointing in opposite directions, and is kept in a feebly stable condition when undisturbed by an external field. On the application of an external magnetic field, which is gradually increased, the orientation of a magnetic particle is at first slightly changed, the change in its moment being approximately proportional to H. Eventually, however, the particle breaks away from its feebly stable direction of equilibrium and may swing (with dissipation of energy) into a new stable state of equilibrium in which it is controlled, perhaps by two other fixed magnets pointing in opposite directions. As the external field increases still further the particle may be pulled away from this direction and be eventually set in alignment with the field.

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[*] The magnetic properties of solids have also been studied recently with the aid of quantum theory. Reference may be made to a paper by P. Ehrenfest, Amsterdam Proc. Vol. 19, (1921), p. 1011, Zeitschr. f. Phys. Bd. 5, (1921), p. 35.

[*] For a discussion of the phenomena of paramagnetism, diamagnetism and ferromagnetism reference may be made to Ewing's book "Magnetic Induction in Iron and other bodies" and to the numerous papers of R. Gans, K. Honda, P. Weiss and other writers. See also Livens. "The Theory of Electricity." S. R. Williams, 'Science," Oct. 14, (1921), p. 339. The subject is treated very fully in the report of the committee on magnetism. Bull. Nat. Research Council Vol. 3, No. 18, 1922.

According to this theory, then, the simple relation can be regarded as approximately correct for weak fields, μ being treated as a constant for the magnetic substance at a given temperature, when the substance is under no stress, etc. In the case of a crystal, however, it was concluded that several constants are needed to express the relation between magnetic force and magnetic induction just as several constants are needed to express the relation between stress and strain. As a result of Plücker's experiments Lord Kelvin suggested the general relations[47]

$$\left.\begin{aligned} B_x &= \mu_{11}H_x+\mu_{12}H_y+\mu_{13}H_z \qquad & \mu_{23}&=\mu_{32} \\ B_y &= \mu_{21}H_x+\mu_{22}H_y+\mu_{23}H_z \qquad & \mu_{31}&=\mu_{13} \\ B_z &= \mu_{31}H_x+\mu_{32}H_y+\mu_{33}\mathrm{H}_z \qquad & \mu_{12}&=\mu_{21} \end{aligned}\right\} \tag{117}$$

Faraday's researches indicated the existence of a dielectric constant for electrical action and the simple relation $D=kE$, which was first introduced, was elaborated in Kelvin's theory of a dielectric, the final equations for a crystal being of the form*

$$\left.\begin{aligned} D_x &= k_{11}E_x+k_{12}E_y+k_{13}E_z \qquad & k_{23}&=k_{32} \\ D_y &= k_{21}E_x+k_{22}E_y+k_{23}E_z \qquad & k_{31}&=k_{13} \\ D_z &= k_{31}E_x+k_{32}E_y+k_{33}E_z \qquad & k_{12}&=k_{21} \end{aligned}\right\} \tag{118}$$

It will be noticed that there is a close analogy in this respect between electric and magnetic phenomena. This analogy is useful for many purposes, but it must not be pushed too far for there are some differences of a quantitative nature. Thus in the theory of a dielectric it is generally assumed that there is a separation of charges of opposite signs under the influence of an electric field, *i. e.*, the moment of an elementary electrical polarized particle may alter in magnitude as well as direction;†

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* In Maxwell's electromagnetic theory of light these equations are assumed to hold approximately also for variable fields and in particular for waves of light over a considerable portion of the long wave spectrum. For an experimental verification see H. Rubens, Zeitschr. f. Phys., Bd. I, (1920), p. 11.

† The hypothesis that the molecules are directed under the influence of an electric field was first used by Langevin in a theory of double refraction. It was afterwards used by Debye in his theory of the dependence of the dielectric constants on temperature. The subject has been discussed at some length by R. Gans, Ann. d. Phys. Bd. 64 (1921), p. 481.

while in the theory of magnetic induction it is generally assumed that the moment of a magnetically polarized particle may be changed in direction but not appreciably in magnitude.

The analogy between magnetism and electricity has been made more complete by the discovery of piezoelectricity and pyroelectricity.

A theory of these phenomena has been developed by W. Voigt[43] with the aid of empirical equations and the form of these equations has been supported by the recent work of Born[44] on the elastic properties of crystal lattices, at least as far as primary effects are concerned.

Voigt expresses the electric moment or polarization produced by given stresses or strains in the form

$$\left.\begin{aligned}
P_x&=e_{11}x_x+e_{12}y_y+e_{13}z_z+e_{14}y_z+e_{15}z_x+e_{16}x_y+\eta_{11}E_x+\eta_{12}E_y+\eta_{13}E_z,\\
P_y&=e_{21}x_x+e_{22}y_y+e_{23}z_z+e_{24}y_z+e_{25}z_x+e_{26}x_y+\eta_{21}E_x+\eta_{22}E_y+\eta_{23}E_z,\\
P_z&=e_{31}x_x+e_{32}y_y+e_{33}z_z+e_{34}y_z+e_{35}z_x+e_{36}x_y+\eta_{31}E_x+\eta_{32}E_y+\eta_{33}E_z,
\end{aligned}\right\}\quad(119)$$

where the coefficients, e_{mn}, are the *piezoelectric constants*, the coefficients η_{mn}, *polarization constants* analogous to those giving the polarization in term of the electric field when there is no strain, P_x, P_y, P_z are the components of the polarization and x_x, y_y, z_z, y_z, z_x, x_y are the six components of strain.

Voigt also expresses the general stress components in the form

$$\left.\begin{aligned}
X_x&=-c_{11}x_x-c_{12}y_y-c_{13}z_z-c_{14}y_z-c_{15}z_x-c_{16}x_y-e_{11}E_x\\
&\qquad-e_{21}E_y-e_{31}E_z\\
Y_y&=-c_{21}x_x-c_{22}y_y-c_{23}z_z-c_{24}y_z-c_{25}z_x-c_{26}x_y-e_{12}E_x\\
&\qquad-e_{22}E_y-e_{32}E_z\\
Z_z&=-c_{31}x_x-c_{32}y_y-c_{33}z_z-c_{34}y_z-c_{35}z_x-c_{36}x_y-e_{13}E_x\\
&\qquad-e_{23}E_y-e_{33}E_z\\
Y_z&=-c_{41}x_x-c_{42}y_y-c_{43}z_z-c_{44}y_z-c_{45}z_x-c_{46}x_y-e_{14}E_x\\
&\qquad-e_{24}E_y-e_{34}E_z\\
Z_x&=-c_{51}x_x-c_{52}y_y-c_{53}z_z-c_{54}y_z-c_{55}z_x-c_{56}x_y-e_{15}E_x\\
&\qquad-e_{25}E_y-e_{35}E_z\\
X_y&=-c_{61}x_x-c_{62}y_y-c_{63}z_z-c_{64}y_z-c_{65}z_x-c_{66}x_y-e_{16}E_x\\
&\qquad-e_{26}E_y-e_{36}E_z
\end{aligned}\right\}\quad(120)$$

where the coefficients c_{mn} are elastic constants.* Substituting the expressions for the strains derived from these equations in the preceding equations the polarization may be expressed directly in terms of the generalized stresses, and the electric force, the coefficients of the stress being the *piezoelectric moduli.*

As in the case of magnetism, these relations cannot be regarded as entirely satisfactory because the phenomena are complicated by a type of hysteresis.* When there is no external electric field the electric field arising from the polarization produced by an applied stress will itself produce a secondary polarization and so we have to distinguish between primary and secondary effects; but the secondary effects are usually small; they have, however, been studied by Voigt in some special problems.[48] The pyroelectricity of a crystal is generally supposed to arise from the strains introduced by a change of temperature; but if a true pyroelectricity is eventually found to exist it may be necessary to add terms proportional to the temperature to equations (119).

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* The *c*'s and *e*'s are generally independent in Born's theory of a crystal, which seems to give a satisfactory foundation for the multiconstant theory of elasticity.

According to Lord Kelvin, W. Voigt and others, a pyroelectric crystal possesses a permanent electric moment and is analogous to a permanent magnet, but under ordinary conditions the external effect of this moment is masked by a compensating effect of a surface charge of electricity which gathers on the surface. The calculations of Voigt[49] indicate that in the case of a crystal without a center of symmetry, it is not possible by means of observations to determine the magnitude of the permanent electric moment because in a deformation the effects of the geometrical and physical changes are added together in such a manner that they cannot be separated. In the case of a crystal with a center of symmetry, the theory requires modification and a separation of the geometrical and physical effects seems possible.

Piezoelectric moduli for certain acentric crystals have been calculated by Born[50] and his co-workers, using the lattice theory; the agreement with the experimental values is, however, not perfect.

In the lattice theory a certain configuration of electric charges, called the base, repeats itself periodically as we go from one cell of the lattice to another. The ordinary elastic strains arise from displacement of each base as a whole, while the piezoelectric strains which give rise to the effective electric moments arise from displacements of the charges in a base relative to each other. A state of initial strain in the base may give a permanent electric moment and it should be noticed that this moment depends on rotational terms of type

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\quad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},$$

u, v, w, being the component displacements, as well as on terms of the type

$$x_x=\frac{\partial u}{\partial x},\quad y_z=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} \tag{121}$$

which represent the components of strain. Voigt has shown, however, that observable phenomena are not influenced by these rotational terms.

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* See, for instance, J. Valasek, Phys. Review, Vol. 17 (1921), p. 475; Vol. 19, (1922), p. 478.

In the case of a crystal with a center of symmetry,[50] it is apparently possible, according to Voigt, to account for the observed phenomena by means of a volume distribution of quadrupoles, the electric potential being given by

$$\Phi=\frac{\partial^2}{\partial x^2}\int\frac{\Pi_{11}}{r}dV+\frac{\partial^2}{\partial y^2}\int\frac{\Pi_{22}}{r}dV+\frac{\partial^2}{\partial z^2}\int\frac{\Pi_{33}}{r}dV+2\frac{\partial^2}{\partial y\partial z}\int\frac{\Pi_{23}}{r}dV$$
$$+2\frac{\partial^2}{\partial z\partial x}\int\frac{\Pi_{31}}{r}dV+2\frac{\partial^2}{\partial x\partial y}\int\frac{\Pi_{12}}{r}dV. \tag{122}$$

The empirical relations adopted as a first approximation are now of the form

$$\left.\begin{aligned}\Pi_{11}&=t_1+t_{11}x_x+t_{12}y_y+t_{13}z_z+t_{14}y_z+t_{15}z_x+t_{16}x_y,\\ \Pi_{23}&=t_4+t_{41}x_x+t_{42}y_y+t_{43}z_z+t_{44}y_z+t_{45}z_x+t_{46}x_y.\end{aligned}\right\} \tag{123}$$

where the t_n's are functions of the temperature alone and the t_{mn}'s are constants.

The phenomena of electrostriction and magnetostriction are of a different order and will not be discussed here. Reference may be made, however, to some recent work on the subject.[51]

In a conducting substance it is necessary to introduce an additional term J into the field equations of a dielectric and to postulate an empirical relation such as Ohm's law

$$J=\sigma E \tag{124}$$

The idea of magnetic conductivity has been used by W. Arkadiew, Phys. Zeitschr. Bd. 14, (1913), p. 928, Ann. d. Phys. Bd. 65 (1921), p. 643, but is not generally adopted.

In a crystal the relation between the conduction current J and the electric force E is generally expressed to a first approximation by the following generalisation of Ohm's law.

$$\left.\begin{aligned}J_x&=\sigma_{11}E_x+\sigma_{12}E_y+\sigma_{13}E_z\\ J_y&=\sigma_{21}E_x+\sigma_{22}R_y+\sigma_{23}E_z\\ J_z&=\sigma_{31}E_x+\sigma_{32}E_y+\sigma_{33}E_z\end{aligned}\right\} \tag{125}$$

where the coefficients σ_{mn} are constants which may be called the *conductivities* of the crystal. In a moving conductor the relation for an isotropic substance seems to be of type[52]

$$J_w-w\rho=\frac{\sigma}{\gamma}\left[E+\frac{1}{c}(w\times B)\right]_w,\quad J_{\bar{w}}=\sigma\gamma\left[E+\frac{1}{c}(w\times B)\right]_{\bar{w}},\quad \gamma=\frac{1}{\sqrt{1-\frac{w^2}{c^2}}} \tag{126}$$

where the suffixes w and $\bar{w}$ are used to indicate components in the direction of w and at right angles to w respectively.

It cannot be said that a theory of this type furnishes a satisfactory theory of electric conduction. It seems better to form a definite picture of the electronic processes and the part played by the atoms and the pressure and the thermal agitation. The recent researches of Bridgman seem to provide us with a good picture and reference may be made to his papers.[53]

The use of empirical equations of the above type is recommended simply as a substitute for more precise relations depending on definite knowledge of the structure of matter and now that the properties of crystals and metals has been revealed to some extent by X-rays, attempts have been made to use the theory of lattice structures in which atoms are at the nodes of lattices and their electrons are capable of vibration. Notable success has already been obtained in this line of work and P. P. Ewald[54] has succeeded in developing a theory of the propagation of waves in a crystal in which, for the case of a long wave-length, the wave surface is a Fresnel wave-surface just as in the ordinary theory based on the empirical relations. A noteworthy feature of this theory and that of Oseen[54] is that the waves in the interior of the crystal arise from free vibrations of electric doublets at the nodes of a lattice. The conditions at the boundary offer some difficulties that are not insuperable. A precise meaning has been given to the incident wave in this theory in a recent paper by Faxén.[55]

The theory of dispersion presents some difficulties on account of the apparent incompatibility of the old theory with the quantum theory of radiation.* The old theory has, however, been developed further with some success[56] and attempts have been made to form a satisfactory theory of the natural optical activity of crystals,[57] a phenomenon that seems to be connected in some way with the piezoelectric properties.

Empirical relations can still be used to advantage in dealing with the optical properties of media when there is dispersion, if it is understood that the optical coefficients are functions of the wave-length. The coefficients may be treated as approximately constant for wave-lengths that are very different from those of the absorption bands of the substance in the case of a crystal; this is generally true in the case of long waves. The empirical relations are useful for the treatment of optical phenomena in moving bodies and in particular for a theory of the propagation of light in moving water in Fizeau's experiment and in the moving crystal in Zeeman's modified form of this experiment.[58]

Many methods of deriving constitutive relations for a moving medium have been based on the transformations of the theory of relativity. The following general method[59] seems to have advantage over the others in directness.

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* The situation has been reviewed recently by P. S. Epstein, Zeitschrift für Physik, Bd. 9, 1922, p. 92. After discussing some of the attempts to partially combine quantum theory with the classical theory he concludes that a satisfactory theory of dispersion must be based on entirely new hypotheses and that, in particular, Bohr's frequency condition must be used.

We commence by writing the electromagnetic equations for a dielectric in the integral form[60]

$$\iint \pm[B_x d(y, z)+B_y d(z,x)+B_z d(x,y)+cE_x d(x,t)+cE_y d(y,t)$$
$$+cE_z d(z,t)]=0 \qquad (127)$$
$$\iint \pm[D_x d(y,z)+D_y d(z,x)+D_z d(x,y)-cH_x d(x,t)-cH_y d(y,t)$$
$$-cH_z d(z,t)]=0 \qquad (128)$$

where the integrals are taken over closed surfaces and t is expressed in terms of x, y and z by means of some arbitrary uniform law. The ambiguity in sign may be removed by choosing the sign so that the expression under the integral sign reduces to an element of an ordinary surface integral when t is constant.

The conditions to be satisfied at the boundary between two adjacent dielectrics are simply that the integrands should be continuous at the boundary, the quantities $d(y, z)$, etc., being derived from increments (dx, dy, dz, dt), $(\delta x, \delta y, \delta z, \delta t)$ which refer to points which always lie on the boundary, whether this is at rest or in motion.

The first integral form may be supposed to vanish for a moving or stationary line of magnetic induction, or for a moving surface made up of such lines, *i.e. a perfect conductor*. The second integral form may be supposed to vanish for a moving line of electric induction or for a moving surface made up of such lines, *i.e. a tube of electric induction*.

The constitutive relations should now indicate a method of expressing one integral form in terms of the other, and conversely if we can derive such a method of deduction from some rule in the absolute calculus[61] we shall have a general method of deriving a scheme of constitutive relations from the differential form or forms with which the rule is associated.[59]

There is a very simple rule associated with a general quadratic differential form

$$Q=\sum_{m,n} g_{mn}dx_m dx_n, \qquad (129)$$
$$(x_1=x, \quad x_2=y, \quad x_3=z, \quad x_4=t)$$

whose coefficients are functions of x, y, z and t. The geometrical analogy is reciprocation with regard to a quadric, the integral forms corresponding to linear complexes of straight lines. The two integral forms are said to be reciprocals when their coefficients are connected by relations of type

$$\left.\begin{aligned} KB_x\sqrt{-g}&=\sum_1^4\sum_1^4 g_{2m}g_{3n}D_{mn}\\ cKE_x\sqrt{-g}&=\sum_1^4\sum_1^4 g_{1m}g_{4n}D_{mn}\end{aligned}\right\} \qquad (130)$$

where

$$\left.\begin{array}{c} D_{mn}=-D_{nm} \\ D_{14}=D_x,\ D_{24}=D_y,\ D_{34}=D_z, \\ D_{23}=-cH_x,\ D_{31}=-cH_y,\ D_{12}=-cH_z. \end{array}\right\} \tag{131}$$

$$g=\begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix} \tag{132}$$

and K is a constant at our disposal.

The vanishing of the quadratic form Q is supposed to indicate the way in which waves of light are propagated through the medium and to give a direct means of determining the velocity along a ray.

In the case of a stationary isotropic medium, the assumption

$$Q=dx^2+dy^2+dz^2-\frac{c^2}{k\mu}dt^2 \tag{133}$$

combined with the choice $K=-\sqrt{\frac{k}{\mu}}$ gives $D=kE,\ B=\mu H.$

When we pass to the case of a uniformly moving isotropic medium by using the relativity transformation

$$x=\frac{x'+vt'}{\beta},\ y=y',\ z=z',\ t=\frac{c^2t'+vx'}{c^2\beta},\ c\beta=\sqrt{c^2-v^2}, \tag{134}$$

we assume that the two integral forms and the quadratic form Q are invariants. We then have

$$Q=c^2\frac{dx'^2+2vdx'dt'+v^2dt'^2}{c^2-v^2}+dy'^2+dz'^2-\frac{c^2}{k\mu}\frac{c^4dt'^2+2vc^2dx'dt'+v^2dx'^2}{c^2(c^2-v^2)} \tag{135}$$

With the same choice of K as before we obtain the constitutive relations

$$\left.\begin{array}{l} B_x=\mu H_x \\ (c^2-v^2)B_y=\left(c^2\mu-\frac{v^2}{k}\right)H_y-vc\left(\mu-\frac{1}{k}\right)D_z \\ (c^2-v^2B_z=\left(c^2\mu-\frac{v^2}{k}\right)H_z+vc\left(\mu-\frac{1}{k}\right)D_y \\ \frac{1}{k}D_x=E_x \\ \left(\frac{c^2}{k}-\mu v^2\right)D_y+vc\left(\frac{1}{k}-\mu\right)H_z=(c^2-v^2)E_y \\ \left(\frac{c^2}{k}-\mu v^2\right)D_z-vc\left(\frac{1}{k}-\mu\right)H_y=(c^2-v^2E_z \end{array}\right\} \tag{136}$$

which are equivalent to those given by Einstein and Laub[52]. When $dy'=dz'=0$, the equation $Q=0$ gives for the ray velocity w

$$w=\frac{dx'}{dt'}=\frac{\pm\frac{c}{\sqrt{k\mu}}\left[1-\frac{v^2}{c^2}\right]+v\left[1-\frac{1}{k\mu}\right]}{1-\frac{v^2}{c^2k\mu}}=\pm\frac{c}{\sqrt{k\mu}}+v\left[1-\frac{1}{k\mu}\right]$$

or, if $\sqrt{k\mu}=n$

$$w=\pm\frac{c}{n}+v\left(1-\frac{1}{n^2}\right) \tag{137}$$

This is the usual equation for Fresnel's dragging coefficient.

The general equations (130) are closely connected with those adopted by Einstein[62] De Donder[63] and Wiechert[64] in their theories of gravitation. De Donder, however, adopts a slightly different definition of magnetic induction and electric force, his B and E, when multiplied by $\sqrt{-g}$ giving our B and cE. The field equations then have the form

$$\left.\begin{aligned} div\ D&=\rho \quad rot\ H=C \\ div\ (B\sqrt{-g})=0,\ rot\ (E\sqrt{-g})&=-\frac{\partial}{\partial t}(B\sqrt{-g}) \end{aligned}\right\} \tag{138}$$

When we adopt the conventional language of Euclidean geometry a gravitational field of Einstein's type is, in our opinion, comparable with a material medium with electric and magnetic polarizations. There is, however, the important distinction that in a gravitational field the coefficients g_{mn} satisfy certain gravitational equations which are characteristic of Einstein's theory and to obtain a material medium whose properties are associated with a quadratic differential form having coefficients of the right type we may have to introduce polarizations which travel with velocity c. Fields arising from polarization of this type will be studied in Section 10.

The above constitutive relations between the vectors B, D, E and H are not sufficiently general to cover the case of a moving crystalline medium which exhibits the phenomenon of double refraction and of course they do not give an account of the phenomena of dispersion and rotary polarization. For greater generality we may introduce, in place of the quadratic differential form, a quadratic integral form

$$I=\sum_{l,m,n,p} g_{lmnp}d(x_l,\ x_m)d(x_n,\ x_p) \tag{139}$$

$$g_{lmpn}=-g_{lmnp},\quad g_{lmnp}=-g_{mlnp}, \tag{140}$$

in which the coefficients g_{lmnp} are functions of the variables x_n.

The integral forms in equations (127-128) are then assumed to be reciprocals with respect to the quadratic integral form when the relations between the vectors are of type

$$\left.\begin{aligned} \lambda B_x &= \sum_{n,p} g_{23np} D_{np} \\ c\lambda E_x &= \sum_{np} g_{14np} D_{np} \end{aligned}\right\} \tag{141}$$

where λ is at our disposal.

For the case of a stationary crystal which does not exhibit piezoelectric properties under the conditions under contemplation, we may use the quadratic integral form

$$\left.\begin{aligned} I = {} & \mu_{11}\{d(y, z)\}^2 + \mu_{22}\{d(z, x)\}^2 + \mu_{33}\{d(x, y)\}^2 + 2\mu_{23}\{d(z, x)d(z, y)\} \\ & + 2\mu_{31}d(x, y)d(y, z) + 2\mu_{12}d(y, z)d(z, x) - c^2[K_{11}\{d(x, t)\}^2 \\ & + K_{22}\{d(y, t)\}^2 + K_{33}d(z, t)^2 + 2K_{23}d(y, t)d(z, t) \\ & + 2K_{31}d(z, t)d(x, t) + 2K_{12}d(x, t)d(y, t) \end{aligned}\right\} \tag{142}$$

in which the coefficients K_{mn} are the co-factors of the constituents k_{mn} in the determinant

$$\begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix}$$

divided by the determinant itself. By means of the relativity transformation (134) this quadratic integral form is transformed into

$$\begin{aligned} I' = {} & \mu_{11}\left\{d(y', z')\right\}^2 + \frac{c^2\mu_{22}}{c^2-v^2}\left\{d(z', x') + vd(z', t')\right\}^2 \\ & + \frac{c^2\mu_{33}}{c^2-v^2}\left\{d(x', y') - vd(y', t'\right\}^2 \\ & + 2\frac{c^2\mu_{23}}{c^2-v^2}\left\{d(z', x') + vd(z', t')\right\}\left\{d(x', y') - vd(y', t')\right\} \\ & + 2\frac{c\mu_{31}}{\sqrt{c^2-v^2}}\left\{d(x', y') - vd(y', t')\right\}d(y', z') \\ & + 2\frac{c\mu_{12}}{\sqrt{c^2-v^2}}\left\{d(z', x') + vd(z', t'\right\}d(y', z') \\ & - c^2\Bigg[K_{11}\left\{d(x', t')\right\}^2 + \frac{K_{22}}{c^2(c^2-v^2)}\left\{c^2d(y', t') - vd(x', y')\right\}^2 \\ & + \frac{K_{33}}{c^2(c^2-v^2)}\left\{c^2d(z', t') + vd(z', x')\right\}^2 \\ & + 2\frac{K_{23}}{c^2(c^2-v^2)}\left\{c^2d(y', t') - vd(x', y')\right\}\left\{c^2d(z', t') + vd(z', x')\right\} \\ & + 2\frac{K_{31}}{c\sqrt{c^2-v^2}}d(x', t')\left\{c^2d(z', t') + vd(z', x')\right\} \\ & + 2\frac{K_{12}}{c\sqrt{c^2-v^2}}d(x', t')\left\{c^2d(y', t') - vd(x', y')\right\} \end{aligned} \tag{142}$$

and we obtain the constitutive relations for the moving crystal in the form

$$\begin{aligned}
B_x' &= \mu_{11}H_x' + \beta\mu_{12}\left(H_y' - \frac{v}{c}D_z'\right) + \beta\mu_{13}\left(H_z' + \frac{v}{c}D_y'\right) \\
B_y' &= \beta^2\mu_{22}H_y' + \beta^2\mu_{23}\left(H_z' + \frac{v}{c}D_y'\right) + \beta\mu_{12}H_x' - \beta^2\frac{v}{c}K_{33}\left(D_z' + \frac{v}{c}H_y'\right) \\
&\quad - \beta^2\frac{v}{c}K_{23}\left(D_y' - \frac{v}{c}H_z'\right) - \beta\frac{v}{c}K_{31}D_z' \\
B_z' &= \beta^2\mu_{33}H_z' + \beta^2\mu_{23}\left(H_y' - \frac{v}{c}\mathrm{D}_z'\right) + \beta\mu_{31}H_x' + \beta^2\frac{v}{c}K_{22}\left(D_y' - \frac{v}{c}H_z'\right) \\
&\quad + \beta^2\frac{v}{c}K_{23}\left(D_z' + \frac{v}{c}H_y'\right) + \beta\frac{v}{c}K_{12}D_z', \\
E_x' &= K_{11}D_x' + \beta K_{12}\left(D_y' - \frac{v}{c}H_z'\right) + \beta K_{13}\left(D_z' + \frac{v}{c}H_y'\right. \\
E_y' &= \beta^2K_{22}D_y' + \beta^2K_{23}\left(D_z' + \frac{v}{c}H_y'\right) + \beta K_{12}D_x' - \beta^2\frac{v}{c}\mu_{33}\left(H_z' + \frac{v}{c}D_y'\right) \\
&\quad - \beta^2\frac{v}{c}\mu_{23}\left(H_y' - \frac{v}{c}D_z'\right) - \beta\frac{v}{c}\mu_{31}H_x' \\
E_z' &= \beta^2K_{33}D_z' + \beta^2K_{23}\left(D_y' - \frac{v}{c}H_z'\right) + \beta K_{31}D_x' \\
&\quad + \beta^2\frac{v}{c}\mu_{22}\left(H_y' - \frac{v}{c}D_z'\right) + \beta^2\frac{v}{c}\left(H_z' + \frac{v}{c}D_y'\right) + \beta\frac{v}{c}\mu_{12}H_x'.
\end{aligned} \tag{143}$$

where

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and the velocity v of the crystal is in the negative direction of the axis of x.

To find the velocity, w, of light along a ray with direction cosines (l, m, n) we write

$$\frac{dx'}{l} = \frac{dy'}{m} = \frac{dz'}{n} = \frac{wdt'}{1} \tag{144}$$

in I' and express the condition that the resulting quadratic form in $\delta x'$, $\delta y'$, $\delta z'$, $\delta t'$ may break up into linear factors. In this way we may obtain the generalised form of Fresnel's dragging coefficient for the case of a moving crystal. An approximate formula for w has been given in an important case by H. A. Lorentz[65].

If, in particular, we write $dx' = udt'$, $dy' = dz' = 0$ and neglect v^2 we obtain the determinantal equation

$$\begin{vmatrix} -c^2K_{11} & K_{12}(c^2+vw) & K_{31}(c^2+vw) \\ K_{12}(c^2+vw) & (w^2+2vw)\mu_{33}-(c^2+2vw)K_{22} & -(w^2+2vw)\mu_{23}-(c^2+2vw)K_{23} \\ K_{31}(c^2+vw) & -(w^2+2vw)\mu_{23}-(c^2+2vw)K_{23} & (w^2+2vw)\mu_{22}-(c^2+2vw)K_{33} \end{vmatrix} = 0 \tag{145}$$

we thus find that

$$w = w_0 - v\frac{A}{B} \tag{146}$$

where w_0 is the velocity along the ray when $v=0$ and

$$\begin{aligned} A = K_{11}[2w_0^2(\mu_{22}\mu_{33}-\mu_{23}^2) - (w_0^2+c^2)(K_{22}\mu_{22}+K_{33}\mu_{33}+2K_{23}\mu_{23}) \\ +2c^2(K_{22}K_{33}-K_{23}^2)] + 2K_{12}K_{31}[(w_0^2+c^2)\mu_{23}-2c^2K_{23}] \\ +2K_{31}^2[(w_0^2+c^2)\mu_{33}-2c^2K_{22}] + 2K_{12}^2[(w_0^2+c^2)\mu_{22}-2c^2K_{33}] \end{aligned} \tag{147}$$

$$\begin{aligned} B = K_{11}[2w_0^2(\mu_{22}\mu_{33}-\mu_{23}^2) - c^2(K_{22}\mu_{22}+K_{33}\mu_{33}+2K_{23}\mu_{23})] \\ +2K_{12}K_{31}c^2\mu_{23}+K_{31}^2c^2\mu_{33}+K_{12}^2c^2\mu_{22} \end{aligned} \tag{148}$$

When

$$K_{11}=K_{22}=K_{33}=K,\ K_{23}=K_{31}=K_{12}=0,\ \mu_{11}=\mu_{22}=\mu_{33}=\mu,$$
$$\mu_{23}=\mu_{31}=\mu_{12}=0,$$

we have

$$\frac{A}{B}=1-\frac{K}{\mu}=1-\frac{1}{n^2}$$

in agreement with the former result due to Fresnel. A geometrical method of deriving the ray-velocity by means of Huygens' construction has been given by A. Anderson[66].

The wave-velocity may be determined by finding the partial differential equation of the characteristics of equations (109) and (143). The wave surface is a type of Kummer's quartic surface* whose equation can be derived from that of Fresnel's surface with the aid of a Lorentz transformation.

It has been shown by Lorentz that

$$w = w_0 - \frac{v_n}{n^2} + v$$

where v_n is the component of the velocity of translation along the ray.

If the medium is a dispersive medium it has been shown by Lorentz that the ray velocity corresponding to (137) is

$$w = \frac{c}{n} \pm v\left(1-\frac{1}{n^2}\right) \mp \frac{v}{n}T\frac{dn}{dt}$$

where T is the period of vibration of the source of light.

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* H. Bateman. Proc. London. Math. Soc. (2), Vol. 8, (1910), p. 375.

It has been shown by Brillouin* that in a dispersive medium energy is propagated with the group velocity so long as the frequencies of light in a plane wave are not too close to those of an absorption band of the substance. The flow of energy has also been discussed in relation to the theory of refraction at a boundary between two substances.

X. FIELDS WITH SINGULARITIES THAT MOVE ALONG STRAIGHT LINES WITH VELOCITY C.

There are some physical phenomena such as the photoelectric effect which seem to indicate that the energy in waves of light is concentrated around points which travel along straight lines with velocity C; and Einstein has, in his theory of the photoelectric effect†, used the hypothesis that energy travels in quanta, a quantum retaining its individuality until it is absorbed as a whole. Einstein's equation for the energy of the electrons emitted in the photo-electric effect has been confirmed by experiment; but physicists are loath to accept the theory by which it was derived, because Einstein's hypothesis seems to be incompatible with the theory of interference.

The quantum theory has, however, come to stay; and one of the outstanding problems of modern theoretical physics is to find some way of bridging the gap between the wave theory of light and the quantum theory. This problem is a difficult one, because some of the results of the two theories seem to be contradictory; thus in the wave theory of light, disturbances diverge from a source in all directions and the intensity of a disturbance diminishes as the distance from the source increases, while the quantum theory seems to require the existence of disturbances which do not spread out but are localised and retain their properties unaltered as they travel along rays. The difficulties may not be insuperable if we may assume, in accordance with the analytical result of Section IV, that there is no radiation to infinity on the whole. Energy that is emitted by one particle must then be absorbed by another; but the new electromagnetic theory depending on the function Ψ has not yet been developed sufficiently to enable one to predict the laws of such emission or absorption.

In the past, many attempts have been made to solve the radiation problem. Sir J. J. Thomson has on many occasions used the idea of discrete tubes of electric force or induction[67] and this idea has been developed by other writers[68]. H. Stanley Allen[69] has recently discussed the properties of discrete tubes of magnetic force in relation to the quantum theory and puts forward the view that magnetic tubes are more fundamental than electric tubes. Sir. W. H. Bragg[70], many years ago, developed an electric doublet theory of radiation which still survives in various forms at the present day. We shall have something to say about this later.

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* Ann. d. Phys. Bd. 44 (1914), p. 203, Comptes Rendus. t. 173 (1921).

† See Hughes' Report. 'Bulletin of the National Research Council,' Vol. 2. April (1921).

If we use a hydrodynamical analogy the emission of radiation may be compared with the production of vortices and Bohr's stationary states of electronic motion may be compared with steady stream line flow in which there is practically no eddy motion. The quantum conditions may be compared with Reynold's criterion for the commencement of turbulence, the angular momentum h being comparable with a Reynolds number LV/ν where ν is the kinematic viscosity.

Pursuing the hydrodynamical analogy, we should look for solutions of the usual field equations

$$\left.\begin{aligned} curl\ H &= \frac{1}{c}\frac{\partial E}{\partial t},\ div\ E=0 \\ curl\ E &= -\frac{1}{c}\frac{\partial H}{\partial t},\ div\ H=0 \end{aligned}\right\} \tag{149}$$

that have singularities different in character from those of the solutions used in the orthodox electromagnetic theory; and our plan will now be to discuss the solutions of these equations from a general point of view. If an attempt were made to solve these equations by the usual methods of the theory of differential equations, the natural plan of attack would be to look for solutions involving arbitrary functions. The first step would be to introduce the complex quantity $Q=H+iE$ so that the equations could be written in the compact form[71]

$$curl\ Q = -\frac{i}{c}\frac{\partial Q}{\partial t},\ div\ Q=0 \tag{150}$$

and then to try to solve these equations by means of a vector Q of form[25]

$$Q=qf(\alpha, \beta) \tag{151}$$

where $f(\alpha, \beta)$ is an arbitrary function of two quantities α and β which are functions of x, y, z and t.

On substituting in the differential equations it is found that Q must have the two forms

$$Q=f(\alpha, \beta)(\nabla\alpha\times\nabla\beta) \tag{152}$$

$$Q=\frac{i}{c}f(\alpha, \beta)\left(\frac{\partial\beta}{\partial t}\nabla\alpha-\frac{\partial\alpha}{\partial t}\nabla\beta\right) \tag{153}$$

Equating these we obtain a vector equation for α and β the general solution of which may be readily obtained by taking x and y as new dependent variables. This solution may be written in the form

$$\left.\begin{aligned} z-ct&=\phi(\alpha,\beta)+(x+iy)\theta(\alpha,\beta)\\ \theta(\alpha,\beta)\,(z+ct)&=\psi(\alpha,\beta)-(x-iy)\end{aligned}\right\} \qquad (154)$$

where θ, ϕ, and ψ are arbitrary complex functions of α and β. These equations define α and β as functions of x, y, z and t. It is clear that there are many types of solution of the desired form.

In particular, if we write

$$\begin{aligned} \phi(\alpha,\beta)&=\zeta(\alpha)-c\tau(\alpha)-[\xi(\alpha)+i\eta(\alpha)]\theta(\alpha,\beta)\\ \psi(\alpha,\beta)&=\theta(\alpha,\beta)[\zeta(\alpha)+c\tau(\alpha)]+\xi(\alpha)-i\eta(\alpha)\end{aligned}$$

the function α is defined by the equation

$$[x-\xi(\alpha)]^2+[y-\eta(\alpha)]^2+[z-\zeta(\alpha)]^2=c^2[t-\tau(\alpha)]^2$$

and β by the equation

$$\theta(\alpha,\beta)=\frac{z-\zeta(\alpha)-c[t-\tau(\alpha)]}{x-\xi(\alpha)+i[y-\eta(\alpha)]}$$

In particular, we may take $\alpha=\tau$ and $\beta=\theta$. The quantity α may then be regarded as the phase of a disturbance which originates at the moving point $[\xi(\tau),\ \eta(\tau),\ \zeta(\tau)]$; it remains constant for a point P which travels away from this moving point with velocity C along a rectilinear path. If we write

$$d\alpha=\frac{\partial\alpha}{\partial x}dx+\frac{\partial\alpha}{\partial y}dy+\frac{\partial\alpha}{\partial z}dz+\frac{\partial\alpha}{\partial t}dt$$

the coefficients of dx, dy, dz and dt, i.e. the derivatives of α remain constant as P moves. This is a consequence of the fact that α satisfies the partial differential equation

$$\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial z}\right)^2=\frac{1}{c^2}\left(\frac{\partial\alpha}{\partial t}\right)^2.$$

The quantity θ which is the ratio of certain linear combinations of the derivatives of α is also a solution of this equation and is a natural type of parameter to use to specify a ray. There is, in fact, one value of θ for each ray through a point $[\xi(\tau),\ \eta(\tau),\ \zeta(\tau)]$ and vice-versa*. The quantities α and θ thus have simple physical meanings; we may call them *phase-parameter* and *ray-parameter* respectively. It is of some interest to notice that the above analysis gives us a method of deriving fields satisfying Maxwell's equations directly from these natural parameters.

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* The real and imaginary parts of θ form with α a set of three independent parameters which enable the entity P to be distinguished from others as in the analysis of Section 1.

Let us consider two sets of independent increments (dx, dy, dz, dt), $(\delta x, \delta y, \delta z, \delta t)$ of the co-ordinates of P and write

$$Y(\alpha, \theta)d(\alpha, \theta) = Q_x d(y, z) + Q_y d(z, x) + Q_z d(x, y) - ic\overline{Q}_x d(x, t) - ic\overline{Q}_y d(y, t) - ic\overline{Q}_z d(z, t)$$

then it is found from the forms of Q and $\overline{Q}$ that $\overline{Q} = Q$ and that Q satisfies equation (150), moreover the ratios of Q_x, Q_y and Q_z depend only on θ and so are constant along a ray.

The equations (150) thus occur simply as equations satisfied by certain auxiliary quantities associated with the ray parameters and it may be that this is the true meaning of Maxwell's equations. The entity that travels along a ray with velocity C may be the primary quantity and we may have to return to a kind of corpuscular theory. We may call the entity a light-corpuscle, or light-particle. But this implies that the corpuscles produce light. It is, perhaps, better to regard the entity simply as an electrical point or an aggregate of such points, using the term electrical point in the sense defined in Section I. In this sense we may speak of an electrical theory of the aether.

The field (152) specified by the functions α, β, Q may also be derived from potentials $C = A - iB$, $\Gamma = \Phi - i\Omega$, by writing

$$Q = H + iE = curl\, C = -\frac{i}{c}\left[\frac{\partial C}{\partial t} + c\nabla\Gamma\right] \tag{155}$$

where

$$C = F(\alpha, \beta)\nabla\theta, \quad c\Gamma = -F(\alpha, \beta)\frac{\partial\theta}{\partial t} \tag{156}$$

$$f(\alpha, \beta) = \frac{\partial(F, \theta)}{\partial(\alpha, \beta)} \tag{157}$$

The quantity θ may be shown to be a solution of the wave equation $\Box\theta = 0$ and we have the relation

$$div\, C + \frac{1}{c}\frac{\partial\Gamma}{\partial t} = 0 \tag{158}$$

In terms of the functions F and θ we have

$$Q = H + iE = \nabla F \times \nabla\theta = \frac{i}{c}\left(\frac{\partial\theta}{\partial t}\nabla F - \frac{\partial F}{\partial t}\nabla\theta\right) \tag{159}$$

It should be noticed that $(Q.Q) = 0$, consequently

$$E^2 - H^2 = 0, \quad (E \cdot H) = 0 \tag{160}$$

We also have the relation $C^2 = \Gamma^2$ which means that

$$A^2 - \Phi^2 = B^2 - \Omega^2 \quad , \quad A \cdot B = \Phi\Omega \tag{161}$$

It can be proved that the above formulae give the general solution of the problem of finding a solution of equations (149) and (160). We shall call an electromagnetic field, whose field vectors satisfy these equations, a *simple radiant field.*

If complex quantities α and β are given arbitrarily, it is not generally possible to determine real values of x, y, z and t which satisfy equations (154); but when a certain relation is satisfied, viz:

$$\phi_2+\theta_2\psi_1-\theta_1\psi_2=0, \quad (\theta=\theta_1+i\theta_2,\ \phi=\phi_1+i\phi_2,\ \psi=\psi_1+i\psi_2 \tag{162}$$

there are ∞^1 possible values of x, y, z, t and these are successive space-time co-ordinates of a point which moves along a straight line with velocity C. We thus have a system of ∞^3 rays associated with the electromagnetic field. It can be shown that Poynting's vector at any point (x, y, z, t) is in the direction of the ray through this point, and electromagnetic energy may be supplied to flow along the rays with velocity C.

To give an example of a simple radiant field we write

$$\alpha=t-\frac{r}{c},\ \beta=\log\frac{z-r,}{x+iy},\ f(\alpha,\beta)=f(\alpha) \tag{163}$$

where r is the distance of the point (x, y, z) from the origin and $f(\alpha)$ is a real function of α. We then have[72]

$$\left.\begin{aligned}
E_x&=-\frac{xz}{r(x^2+y^2)}f\left(t-\frac{r}{c}\right) & H_x&=\frac{y}{x^2+y^2}f\left(t-\frac{r}{c}\right)\\
E_y&=-\frac{yz}{r(x^2+y^2)}f\left(t-\frac{r}{c}\right) & H_y&=-\frac{x}{x^2+y^2}f\left(t-\frac{r}{c}\right)\\
E_z&=\frac{1}{r}f\left(t-\frac{r}{c}\right) & H_z&=0
\end{aligned}\right\} \tag{164}$$

If f(α) is zero except during a very short interval of time, $\delta\alpha$, the field is clearly zero except within the thin expanding spherical shell for which $\alpha=t-\dfrac{r}{c}$ has values for which $f(\alpha)\pm 0$. The field has isolated singularities which travel along the axis of z with velocity C in opposite directions, the singularities being radiated from the origin.

These singularities carry electric charges which are equal and opposite in sign as may be seen by integrating the normal electric force over the plane $z=0$. The total electric charge associated with one singularity is [25]

$$2\pi c\int_{-\infty}^{\alpha_1} f(\alpha)\,dx=2\pi c\int_{\alpha_1}^{\alpha_2} f(\alpha)d\alpha$$

where $\alpha_1<\alpha<\alpha_2$ is the range of values of α for which $f(\alpha)\pm 0$.

If we write down the usual expressions for the vectors E and H in the field of an electric pole which moves with uniform velocity along a straight line we have expressions of the following type for the components at time t.

$$\left.\begin{array}{lll} E_x=(1-\gamma^2)\dfrac{eX}{s^3}, & E_y=(1-\gamma^2)\dfrac{eY}{s^3}, & E_z=(1-\gamma^2)\dfrac{eZ}{s^3}, \\ H_x=0 & H_y=-\gamma E_z, & H_z=\gamma E_y \end{array}\right\} \quad (165)$$

where $(x.y.z)$ are the co-ordinates of a point relative to the position of the pole at time t, $c\gamma$ is the velocity of the pole in the direction OX and

$$s=\sqrt{x^2+(1-\gamma^2)(y^2+z^2)} \qquad (166)$$

Now when γ^2 approaches 1 the field tends to become evanescent except in the plane $X=0$ transverse to the direction of motion and in this plane E_y and E_z become infinite. In the field considered above, however, we had electric charges travelling with the velocity of light, and the field did not become infinite at all points of two planes perpendicular to the direction of motion.

The explanation of this paradox seems to be this: An electric charge does not *produce* a field at all *while* it travels along a straight line with velocity C. Whatever field is associated with such a charge must be supposed to have been produced when the charge was separated from a compensating charge of opposite sign or when the direction of motion of the charge was changed by a collision. The nature of this field depends, moreover, upon the manner in which the charge was separated from its fellow, i. e., the manner in which it was "created" so to speak.

In the case of the field (165) the charge must be supposed to have been created suddenly an infinitely long time ago, i.e. at time $t=-\infty$. The thin spherical shell of our former example thus becomes an infinitely thin plane shell and $f(\alpha)$ is infinite while its integral from α_1 to α_2 is finite, $\alpha_1-\alpha_2$ being infinitesimal.

In the type of field represented by the expressions (164) the charges are created more gradually, the spherical shell is not infinitely thin and this is why the electric force is not infinite all over the shell. Another peculiarity of a field of type (164) is that it may have a beginning and an end.

These properties of fields of type (152) suggest that such fields may throw some light on the nature of the aether. If the aether is supposed to be made up of electricity travelling along straight lines with velocity C, the electromagnetic fields may be produced by collisions between the aether particles. An aether particle may consist normally of an electric doublet travelling with velocity C and its constituents, according to the above idea, must be supposed to be separated gradually and not suddenly.

Simple radiant fields may be generalised either by integration or differentiation, both of these being particular cases of the general process of superposition. One method of generalisation by integration is to make the functions θ, ϕ, ψ and f functions of a parameter ω and to define a new vector $\bar{Q}$ by the equation[73]

$$\bar{Q}=\int_{\omega_1}^{\omega_2} Q d\omega \tag{166}$$

where ω_1 and ω_2 are either constants or functions of $x.y.z.t$ which satisfy equations of type

$$G_1(\alpha, \beta, \omega_1)=0, \qquad G_2(\alpha, \beta, \omega_2)=0 \tag{167}$$

G_1 and G_2 being arbitrary functions with continuous derivatives. This method of generalisation is useful for the solution of diffraction problems.[74]

Another method of generalisation may be described as follows:—
Let Q_s denote the vector Q when we use the function $f_s(\alpha, \beta)$ instead of $f(\alpha, \beta)$, then we may define a complex field of the first order by means of the equation

$$Q'=\sum_s \frac{\partial}{\partial x_s} Q_s, \tag{168}$$

use being now made of the notation $x_1=x$, $x_2=y$, $x_3=z$, $x_4=t$. Again, if Q_{mn} denotes the vector Q when the function $f_{mn}(\alpha, \beta)$ is used instead of $f(\alpha, \beta)$, we may define a complex field of the second order by means of the equation

$$Q''=\sum_m \sum_n \frac{\partial^2}{\partial x_m \partial x_n} Q_{mn} \tag{169}$$

and so on.

To give an example of a field of the first order let us make f constant in equations (164) and differentiate with respect to z. We obtain

$$E_x=-\frac{fx}{r^3}, \quad E_y=-\frac{fy}{r^3}, \quad E_z=-\frac{fz}{r^3}, \quad H=0. \tag{170}$$

This is the field of an electric pole at the origin. If f were not constant we should obtain the field of a pole with a varying electric charge[25] at the origin, electric charges of the same sign being projected in opposite directions from the pole at each instant. The field is in this case

$$\left.\begin{aligned}
E_x&=-\frac{x}{r^3}f\left(t-\frac{r}{c}\right)+\frac{xz^2}{cr^2(x^2+y^2)}f'\left(t-\frac{r}{c}\right),\\
H_x&=-\frac{yz}{cr(x^2+y^2)}f'\left(t-\frac{r}{c}\right)\\
E_y&=-\frac{y}{r^3}f\left(t-\frac{r}{c}\right)+\frac{yz^2}{cr^2(x^2+y^2)}f'\left(t-\frac{r}{c}\right),\\
H_y&=\quad\frac{xz}{cr(x^2+y^2)}f'\left(t-\frac{r}{c}\right)\\
E_z&=-\frac{z}{r^3}f\left(t-\frac{r}{c}\right)-\frac{z}{cr^2}f'\left(t-\frac{r}{c}\right),\qquad H_z=0
\end{aligned}\right\}\tag{171}$$

By superposing a simple radiant field on the field just specified we can get rid of the charges projected in one of the two directions and obtain a field, which, after changing the sign of *f* can be written in the form

$$\left.\begin{aligned}
E_x&=\frac{x}{r^3}f\left(t-\frac{r}{c}\right)+\frac{xz}{cr^2(r+z)}f'\left(t-\frac{r}{c}\right), & H_x&=-\frac{yr}{cr^3(r+z)}f'\left(t-\frac{r}{c}\right)\\
E_y&=\frac{y}{r^3}f\left(t-\frac{r}{c}\right)+\frac{yz}{cr^2(r+z)}f'\left(t-\frac{r}{c}\right), & H_y&=\frac{xr}{cr^3(r+z)}f'\left(t-\frac{r}{c}\right)\\
E_z&=\frac{z}{r^3}f\left(t-\frac{r}{c}\right)-\frac{r-z}{cr^2}f'\left(t-\frac{r}{c}\right), & H_z&=0.
\end{aligned}\right\}\tag{172}$$

This is the field of an electric pole whose associated electric charge varies on account of the emission of electric charges in one direction.[75]

These examples show that by means of simple radiant fields we can cancel out some or all of the singularities of fields of the first order. It can also be shown that some singularities, that move along straight lines with velocity *c*, can be cancelled out by superposing simple radiant fields on fields of the second order. This means, of course, that we can build up fields in which singularities moving with velocity *c* can be either absorbed or deflected.* These singularities may have electric or magnetic charges associated with them or they may be electric or magnetic doublets. It should be mentioned that it is possible to write down a field in which secondary singularities are projected from a primary singularity in directions which vary in an arbitrary manner with the time.[25] The primary singularity, too, may move in an arbitrary manner with a velocity which is always less than *c*.

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* It is thought that this result may have some bearing on Bragg's theory of radiation if this is revived in some form. The author feels inclined to adopt the view that these moving singularities are directly responsible for body forces and, in particular, for the electromagnetic force per unit volume mentioned on p. 101. If this view be adopted, it may be advantageous to change the signs of electromagnetic energy and momentum and limit only *material* energy to be positive.

By suitably choosing the law of variation of projection of the secondary singularities, it is possible to make the locus of the secondary singularities a line of electric force of a moving electric pole.[30] If, moreover, we superpose a number of fields of this type it is possible, at least in some cases, to make the field zero outside a number of discrete tubes of electric force of a moving electric pole. In this way it would be possible to derive a theory of discrete Faraday tubes from Maxwell's equations but it is doubtful whether the theory would be of any physical interest.

The fact that a secondary singularity retains its properties unaltered (in some of the simplest fields at least) while it moves with velocity c along a straight line, suggests that it may be possible to base a satisfactory theory of radiation on Maxwell's equations but we are quite unable to say at present what form this theory should take. There are many difficulties.

If, for instance, we suppose that the secondary singularities carry electric charges and we regard them as symbols for discrete electric charges of small but finite size we must account for the generation of these charges. The first possibility that occurs to one is that the charge of an electron may not be always the same but may depend on the magnitude of the function Ψ. This seems reasonable because it appears from the analysis of Section III that an electron cannot move continuously without loss of charge from a region where the potential Ψ has a value Ψ_0, which is practically constant, to another region where the potential Ψ_0 has another value Ψ_0 which is practically constant. There are in fact four conditions to be satisfied in order that the continuous motion may be possible and there are only three available constants.

Another possibility is that the emitted charges may be derived from electric doublets already existing in the æther and travelling with velocity c along straight lines. An apparent difficulty occurs here because work is needed to separate the component charges of a doublet. It should be pointed out, however, that if the electric field of an electric pole is not exactly the same in all directions—and this is true when the pole has an accelerated motion—it is possible for the work done on one radiated constituent of a doublet against the field of the pole to be more than balanced by the energy gained in separating the other constituent of the doublet from the pole.

It is interesting to inquire whether the radiation from a moving electric pole can be cancelled out in practically all directions by superposing on it fields of type (152) in which secondary singularities are fired out from the moving pole with velocity c.

We have seen that the electric and magnetic vectors in the field of the moving pole can be written in the form

$$H_x=\frac{\partial(\sigma, \tau)}{\partial(y, z)}, \quad E_x=\frac{1}{c}\frac{\partial(\sigma, \tau)}{\partial(x, t)}, \tag{173}$$

where σ and τ are defined by the equations (33), (35) and (36). The radiation depends only on the terms of order $\frac{1}{r}$ and these may be written in the form

$$H_x^*=\frac{\partial(\sigma^*, \tau)}{\partial(y, z)}, \quad cE_x^*=\frac{\partial(\sigma^*, \tau)}{\partial(x, t)} \tag{174}$$

where σ^* is what σ becomes when we omit the term c^2-v^2 in K. To get rid of the radiation we must completely cancel out these terms of order $\frac{1}{r}$. Now a simple radiant field in which secondary singularities are radiated from the moving pole is given by formulae of type

$$H_x+iE_x=\frac{\partial(\lambda, \tau)}{\partial(y, z)}=\frac{i}{c}\frac{\partial(\lambda, \tau)}{\partial(x, t)} \tag{175}$$

where $\lambda=G(\beta, \tau)$ and

$$\beta=\frac{z-\zeta(\tau)-c(t-\tau)}{x-\xi(\tau)+i[y-\eta(\tau)]}. \tag{176}$$

The terms in E and H are now all of order $\frac{1}{r}$ but they will not generally cancel E^* and H^* except, perhaps, when the secondary singularities form a continuous tube or fill the whole of space or a part of it. If E^* and H^* can be cancelled out in any region of space the resultant field will be of type [25]

$$H=\nabla\omega\times\nabla\tau, \quad E=\frac{1}{c}\left[\frac{\partial\tau}{\partial t}\Delta\omega-\frac{\partial\omega}{\partial t}\Delta\tau\right] \tag{177}$$

where

$$\omega=e\frac{c^2-\xi'^2-\eta'^2-\zeta'^2}{(x-\xi)\xi'+(y-\eta)\eta'+(z-\zeta)\zeta'-c^2(t-\tau)} \tag{178}$$

This field reduces to an electrostatic field of the ordinary type when the primary singularity is stationary.

There is a possibility that the terms E^* and H^* may be cancelled out by superposing on the field of the electric pole not only a combination of simple radiant fields, having the pole as primary singularity, but also a combination of fields of the first, second and higher orders with the pole as primary singularity. It is possible, indeed, to cancel out the field of the pole altogether and there is a danger that when we cancel out the radiation terms this is just what will happen. There will, however, be a generalization of (177) which reduces to a combination of the field of an electric pole, dipole, quadrupole, etc., when the primary singularity is stationary; the field reducing to a dipole's field when $v=0$ may be obtained from (177) by differentiating with regard to x, y, z, multiplying the results by arbitrary constants and adding.

In a field of type (177) there are no terms of order $\frac{1}{r}$ in the expressions for E and H, consequently there is no radiation of energy to infinity as the flux represented by Poynting's vector is concerned. There is, however, a flow of electricity to infinity in the case of an accelerated motion of the primary singularity, electricity being projected from this moving point in all directions and travelling along straight lines with the velocity of light. The electric charge associated with the pole, however, remains constant and so the emitted positive electricity must be just balanced by the emitted negative electricity. The field may thus be regarded as being produced by the breaking up of electric doublets at the primary singularity and since there is no radiation of energy to infinity on the whole as represented by the Poynting flux the energy furnished by the repulsions between charges of like sign must be sufficient to balance the work done in separating the charges of opposite signs. This may not, however, be right because it may be necessary to associate mass and energy of non electromagnetic type with the electricity that flows to infinity and then this type of field would not furnish a case of no radiation. The question as to whether it is necessary to associate electromagnetic mass or a mass of other type with an electric charge travelling with velocity c is a difficult one to answer; so much depends on the way in which the charge is generated by a process of electric separation. It is usual to say that an electron would have an infinite mass if it could travel along a straight line with the velocity of light, but this may be because the field of the electron is regarded as the limiting case of the field obtained by superposing fields of point charges of Lienard's type and we have seen that in such fields the field vector E becomes infinite all over a plane when the velocity is c and the motion is rectilinear. In a simple radiant field, on the other hand, we have electric charges moving with velocity c and the field vectors are not infinite except at the electric charges. If, then, we superpose simple radiant fields so as to obtain a volume distribution of such electric charges moving with velocity c, it may not be necessary to associate an *infinite* mass with these charges, but it may still be necessary to associate a finite mass with them. This is a point on which the author feels undecided. If a finite mass must be associated with these electric charges we have a radiation of matter to infinity in a field of type (177) and this is just as bad as a radiation of energy given by the Poynting flux.

It appears then that the solution of the problem of no radiation suggested in Section IV is preferable to the one given here. The field (177) is a particular case of a more general field specified by the equation[76]

$$H+iE=\nabla\theta\times\nabla\tau+\frac{i}{c}\left[\frac{\partial\tau}{\partial t}\nabla\theta-\frac{\partial\theta}{\partial t}\nabla\tau\right]. \tag{179}$$

where

$$\theta=\frac{\alpha(x-\xi)+\beta(y-\eta)+\gamma(z-\zeta)+\delta}{(x-\xi)\xi'+(y-\eta)\eta'+(z-\zeta)\zeta'-c^2(t-\tau)} \tag{180}$$

and α, β, γ and δ are arbitrary complex functions of τ. When δ is real, the lines of electric force in such a field may be obtained by solving a Riccatian equation. When δ is imaginary, the lines of magnetic force may be obtained in a similar manner.

Fields of this type have been studied by the author and by Leigh Page.[77] The latter has used them as a basis of a theory of radiation and has invented a new type of electron whose surface is the locus of the primary singularities of fields of this type, the fields being of such a nature that the electric vectors cancel out in the interior of a stationary electron, while the magnetic vectors cancel out in the external region leaving an electrostatic field of the ordinary type. Page calls his primary fields rotating electric fields and finds that in a complete rotation the energy radiated is almost exactly $\frac{1}{32}$ hν, where ν is the frequency of rotation and h is Planck's constant. In an electromagnetic field of Page's type, magnetism is radiated from the primary singularity and travels along straight lines with velocity c.

Many physicists are of the opinion that Planck's constant h is connected in some way with magnetism. In his work on the magneton or ring-electron[78] S. B. McLaren found that the angular momentum is proportional to the number of tubes of magnetic induction linked through the ring's aperture and to the number of tubes of electric induction terminating on its surface. This result has been recently confirmed and developed by H. Stanley Allen[79] who points out some interesting connections with quantum theory and extends the result to the case in which any number of point charges are revolving round an axis with a common angular velocity. Allen shows that when quantum theory indicates that the mean value of the kinetic energy of an electron describing a periodic orbit with frequency $\frac{1}{\nu}$ is equal to $\frac{1}{2}nh\nu$, this may be regarded also as $\frac{1}{2}Ne\nu$ where $N=n\frac{h}{e}$ is the number of magnetic tubes associated with the moving charge and passing through its orbit. This is an extension of a result given by A. L. Bernoulli.

A connection between quantum theory and magnetism is also suggested by the hydrodynamical analogy, for magnetism is to some extent analogous to eddy motion in an electrical aether.

In the solution by Helmholtz[79] and Lord Kelvin[80] of d'Alembert's hydrodynamical paradox of the absence of resistance in the continuous potential flow of a fluid round an obstacle, it was pointed out that if the pressure p vanishes at any point, then the continuous flow must cease and vortices or some type of discontinuity must form. It is interesting to inquire whether there is any function analogous to the pressure in the case of an electromagnetic field.

Let us write

$$c[E\times H]=qW=\tfrac{1}{2}q(E^2+H^2)$$

and regard q for the moment as analogous to a velocity. This analogy must not be carried too far, for, as Dr. Epstein has remarked to me, q does not generally[*] transform like a velocity in the transformations of the theory of relativity. We have the identity

$$W^2(c^2-q^2)=\frac{c^2}{4}\left[(E^2-H^2)^2+4(E\cdot H)^2\right]=c^2P^2 \qquad (181)$$

where P is Cunningham's principal stress.[†]

Writing $W=\rho c^2$, where ρ is a kind of mass density, we have an equation

$$\frac{\partial\rho}{\partial t}+div\ (\rho q)=0 \qquad (182)$$

analogous to the hydrodynamical equation of continuity and if we neglect squares and higher powers of q^2/c^2 we obtain from (181) an equation

$$\frac{P}{\rho}+\frac{1}{2}q^2=constant \qquad (183)$$

analogous to Bernoulli's integral. The quantity P is thus in some respects analogous to the pressure in hydrodynamics.[81]

P is always greater than zero in the case of the field of a moving electric pole but in the case of the field of two electric poles this is not always the case. Let us consider for instance the case of a Hertzian dipole. The field vectors in polar co-ordinates are[82]

$$\left.\begin{aligned}
E_r&=\frac{2\cos\theta}{r^2}\left\{f'(ct-r)+\frac{1}{r}f(ct-r)\right\}\\
E\theta&=\frac{\sin\theta}{r}\left\{f''(ct-r)+\frac{1}{r}f'(ct-r)+\frac{1}{r^2}f(ct-r)\right\}\\
H\phi&=\frac{\sin\theta}{r}\left\{f''(ct-r)+\frac{1}{r}f'(ct-r)\right\}
\end{aligned}\right\} \qquad (184)$$

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[*] It does transform like a velocity in the case when $q^2=c^2$.

[†] E. Cunningham, "The Principle of Relativity," Ch. XV.

When $\theta=\frac{\pi}{2}$ we have $(E\cdot H)=0$ and $E^2-H^2=0$ whenever $f(ct-r)=0$. There are thus places where $P=0$.

It seems likely also that P will vanish in the case of Bohr's hydrogen atom when the charges are treated as point charges, for $(E\cdot H)=0$ in the plane of the orbit and there are points at which the electric vectors due to the two charges nearly cancel and at such points H may be as big as E and may make $E^2-H^2=0$.

An attempt to develop a hydrodynamical theory of the electromagnetic field has been made by A. Korn.[83] He uses the idea of pulsating spheres in a practically incompressible fluid and adopts a modification of the Principle of d'Alembert. Korn's equations for the aether are of type

$$\left.\begin{aligned} \mathit{curl}\ v_2 &= \frac{1}{c}\left[\frac{\partial v_1}{\partial t}+(v_0\cdot\nabla)v_1-(v_1\cdot\nabla)v_0\right], \quad \mathit{div}\ v_1=0 \\ \mathit{curl}\ v_1 &= -\frac{1}{c}\left[\frac{\partial v_2}{\partial t}+(v_0\cdot\nabla)v_2-(v_2\cdot\nabla)v_0\right], \quad \mathit{div}\ v_2=0 \end{aligned}\right\} \tag{185}$$

where the vector v_0 may be interpreted as the *visible* velocity of the fluid and v_1, v_2 as coefficients which occur in the expression of the actual instantaneous velocity in the form

$$v=v_0(x,y,z,t)+v_1(x,y,z,t)\ \mathit{Cos}\,\frac{2\pi t}{T}+v_2(x,y,z,t)\ \mathit{Sin}\,\frac{2\pi t}{T}. \tag{186}$$

Korn's theory has been developed in detail but has not yet thrown any light on the difficulties presented by the quantum theory.* It should be noticed that the above equations for the aether are slightly different from the usual Maxwell equations.

J. H. Jeans[84] has expressed the opinion that the equations for the aether may really need modification and that the exact equations may involve the fundamental constants e and h of experimental physics.

Modified forms of the equations for the aether actually occur in the theories of gravitation of Einstein, Wiechert and others. According to Reissner,[85] however, the field of an electric pole in the Einstein theory differs only slightly from the field of an electric pole in the ordinary theory. Something more radical seems to be needed to account for the peculiarities of the quantum theory of radiation. Everything seems to point to the conclusion that it is the stress-energy tensor which needs modification and this involves a modification of the force equation.

Meg Nad Saha[32] has proposed a new law of electromagnetic force which is regarded favorably by A. C. Crehore,[88] who has used it in atomic theory. Another law of force has already been suggested in Section II.

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* In two recent notes. Phys. Zeitschr. Bd. 20 (1919), p. 491, Bd. 21 (1920), p. 97. Korn does, however, propose a theory of line spectra based on his idea of pulsating spheres and indicates a possible connection with quantum theory.

From the point of view of Einstein's theory of gravitation, all the foregoing analysis is simply an approximation and light does not travel along straight lines with velocity c in a gravitational field if we still use the language of Euclidean geometry.

Einstein's theory indicates that greater force of expression may be attained by using the language of non-Euclidean geometry. It is, however, still advantageous for some purposes to use the familiar language of Euclidean geometry and then the question arises as to how a gravitational field should be described in terms of familiar entities.

In Einstein's theory, if suitable interpretations are given to the vectors, the electromagnetic equations for the aether are of type

$$\left.\begin{aligned} curl\, H &= \frac{1}{c}\frac{\partial D}{\partial t}, \quad div\, D = 0 \\ curl\, E &= -\frac{1}{c}\frac{\partial B}{\partial t}, \quad div\, B = 0 \end{aligned}\right\} \qquad (187)$$

where D, H, B and E are connected by equations of type (138) in which the coefficients g_{mn} are Einstein's gravitational potentials defined by means of an invariant quadratic form Q. Now if we use the ordinary language of Euclidean geometry these equations imply the existence of electric and magnetic polarizations in the aether and there seems no reason why an Einsteinian electromagnetic field should not be built up by superposition from fields which satisfy Maxwell's equations. There may, however, be this difference between Einstein's aether and an ordinary ponderable medium. Whereas in the case of a ponderable medium we need use only fields in which electric charges move with velocities numerically less than c, to obtain a field in Einstein's aether it may be necessary to introduce fields with singularities which move with velocity c. It seems likely then that electromagnetic fields in Einstein's aether can be built up from simple radiant fields and fields of the first, second and higher orders. In this connection it may be noticed that we may already have a tensor of the first rank associated with a field of the first order and a tensor of the second rank associated with a field of the second order, viz. the tensors defined by the functions f_m, f_{mn}. The definition of a gravitational tensor in terms of electrical quantities does not then seem impossible; indeed it is possible that gravitation may be connected in some way with a variation of the electronic or nuclear electric charge.

In deriving electromagnetic equations of Einstein's type from fields satisfying the usual electronic equations we can eliminate the currents and space charge in the æther by introducing a volume polarization and abandoning the simple relations $D=E$, $B=H$. To illustrate this let us consider the field

$$B=\nabla\sigma_0\times\nabla\tau\quad,\quad E=\frac{1}{c}\left[\frac{\partial\tau}{\partial t}\nabla\sigma_0-\frac{\partial\sigma_0}{\partial t}\nabla\tau\right]\tag{188}$$

where

$$\sigma_0=\frac{(x-\xi)\alpha+(y-\eta)\beta+(z-\zeta)\gamma+c^2-v^2}{M}\tag{189}$$

the notation being that of Section 4, α, β and γ being arbitrary real functions of τ.
We have in this case

$$\mathit{div}\ E=-\frac{k}{c}\frac{\partial\tau}{\partial t},\qquad \mathit{curl}\ B-\frac{1}{c}\frac{\partial E}{\partial t}=k\nabla\tau\tag{190}$$

where k is a complicated function of x, y, z and t whose exact form does not at present interest us. If now we write

$$H=\nabla\sigma_1\times\nabla\tau,\qquad D=\frac{1}{c}\left[\frac{\partial\tau}{\partial t}\nabla\sigma_1-\frac{\partial\sigma_1}{\partial t}\nabla\tau\right]\tag{191}$$

where

$$\sigma_1=\frac{(x-\xi)\xi''+(y-\eta)\eta''+(z-\zeta)\zeta''+c^2-v^2}{M}\tag{192}$$

we have

$$\mathit{curl}\ H=\frac{1}{c}\frac{\partial D}{\partial t},\qquad \mathit{div}\ D=0\tag{193}$$

$$\left.\begin{array}{lll} B=H+q, & D=E+p, & q=-(s\times p)\\ p=\frac{1}{c}\left[\frac{\partial\tau}{\partial t}\nabla\sigma_2-\frac{\partial\sigma_2}{\partial t}\nabla\tau\right], & & \sigma_2=\sigma_1-\sigma_0\\ q=-\nabla\tau\times\nabla\sigma_2, & & \end{array}\right\}\tag{194}$$

s being a unit vector having the same direction as $-\nabla\tau$. It is easy to see that the polarization p is everywhere at right angles to the ray, i. e. to s, and must be regarded as travelling with the velocity of light. The preceding equations may, in fact, be compared with the corresponding equations for a moving dielectric.

In conclusion it may be remarked that a field differing only slightly from a field satisfying Maxwell's equations for the aether may be obtained by averaging the field vectors for any given set of moving charges taking c as a quantity which varies within a narrow range from c_1 to c_2. In such a field there may be no radiation of energy to infinity.

XI. DYNAMICAL EQUATIONS OF MOTION

It is generally believed that electrodynamics can be based on the well known principles of classical mechanics but mathematicians reserve the privilege of being allowed to alter these if necessary to make electrodynamics agree with the quantum theory of radiation.

The general dynamical argument in favour of a dynamical theory of the electromagnetic field was first formulated by Maxwell[87] for the case of the field surrounding a system of linear conductors. His analysis was subsequently extended to cover the more general case, firstly by von Helmholtz[88] and Lorentz,[89] later by Larmor,[90] Macdonald,[91] Abraham[3] and others.[42] In most of the investigations use is made of a Principle of Least Action in the simple form

$$\delta\int_{t_1}^{t_2} L\,dt = \delta\int_{t_1}^{t_2}(T-W)\,dt=0 \qquad (195)$$

wherein T denotes the kinetic energy and W the potential energy of the system in any configuration. The principle is supposed to be formulated in terms of any co-ordinates that are sufficient to specify the configuration in accordance with its known properties and connexions and the variation is supposed to refer to a fixed time of passage of the system from the initial to the final configuration. In the more recent investigations of Livens[92] it is recognized that this simple form of Hamilton's principle involves a complete knowledge of the constitution of the system and that before it can strictly be applied it is necessary to know the exact values of the kinetic and potential energies expressed properly in terms of co-ordinates and velocities. As, however, we have to deal with a system whose ultimate constitution is wholly or partly unknown it is necessary to use a modified form of the principle allowing for a possible ignorance of the constitution of the system.

Suppose that the Lagrangian function is expressible in terms of a certain number of variables $x_1, x_2 \cdots x_k$, which are known to be connected with the chosen co-ordinates q and their velocities q by a series of relations of type

$$M_s=0 \qquad (196)$$

M_s being a function of the co-ordinates q, the velocities $\dot{q}$, the variables x and the differential coefficients of these latter variables with respect to the time. The usual method of procedure is to introduce a set of multipliers λ_s, functions of the time, and then to consider the variations of the integral

$$\int_{t_1}^{t_2}(L+\Sigma\lambda_s M_s)dt \qquad (197)$$

where the q's and x's undergo independent variations in the case when only the first derivatives appear in our expression under the integral sign.

The variational principle then leads to an equation of type

$$\frac{d}{dt}\left(\Sigma_s \lambda_s \frac{\partial M_s}{\partial x}\right) - \Sigma \lambda_s \frac{\partial M_s}{\partial x} - \frac{\partial L}{\partial x} \tag{198}$$

for each variable x and an equation of type

$$\frac{d}{dt}\left(\Sigma \lambda_s \frac{\partial M_s}{\partial q}\right) - \Sigma \lambda_s \frac{\partial M_s}{\partial q} = 0 \tag{199}$$

for each q co-ordinate.

By applying this method Livens has given a very complete and general presentation of the dynamical argument when gravitational actions are not considered in detail as in Einstein's theory. His results are quite compatible with equations of the type considered in Section VIII. An interesting feature of the method is that the electromagnetic potentials are introduced as coefficients of type λ.

The most important result of this type of dynamical reasoning is that the force F exerted by an electromagnetic field on an electric charge, e, moving with velocity v can be represented with considerable accuracy by the vector

$$e\left[E + \frac{1}{c}(v \times B)\right] = F, \tag{200}$$

a correction being necessary, perhaps, if there is radiation of energy, to allow for the reaction of the radiation. A correction may be necessary, too, when the charge is surrounded by a distribution of electric doublets, as was pointed out by Lorentz[93] in his theory of refraction and dispersion.

The Lagrangian function has been calculated for some systems of electric charges and some interesting results have been obtained.[94]

A dynamical investigation of a somewhat different character that is worthy of notice is that of E. Guillaume,[95] who derives the electromagnetic equations with the aid of Appell's general dynamical equations.*

The variational method was extended by G. Mie[9] in his theory of matter and has been used with great success by Einstein,[96] De Donder,[63] Lorentz[96] and Hilbert[97] in Einstein's theory of gravitation. It has also been used by Weyl[98] in his extension of Einstein's theory.

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* These equations are applicable to non-holonomic systems. The peculiarities of the quantum theory of radiation suggest that an electron describing a non-radiating orbit is in some respect analogous to a rolling sphere and an electron which is radiating energy to a sliding sphere. But the analogy with a ball that sometimes rolls and sometimes slides is not of much use.

De Donder expresses the variational principle in the general form

$$\delta\int(a+bC+\Lambda)\sqrt{-g}\,dx_1dx_2dx_3dx_4=o \tag{201}$$

where a and b are universal constants, g is defined by equation (130). C is given by

$$C=\sum_{\alpha}\sum_{\beta}g^{\alpha\beta}G_{\alpha\beta} \tag{202}$$

$$G_{\alpha\beta}=\sum_{\sigma}\sum_{\tau}\left[\frac{\partial}{\partial x_\beta}\begin{Bmatrix}\alpha\sigma\\ \sigma\end{Bmatrix}-\frac{\partial}{\partial x_\sigma}\begin{Bmatrix}\alpha\beta\\ \sigma\end{Bmatrix}+\begin{Bmatrix}\beta\tau\\ \sigma\end{Bmatrix}\begin{Bmatrix}\alpha\sigma\\ \tau\end{Bmatrix}-\begin{Bmatrix}\sigma\tau\\ \sigma\end{Bmatrix}\begin{Bmatrix}\alpha\beta\\ \tau\end{Bmatrix}\right] \tag{203}$$

where

$$\begin{Bmatrix}\alpha\beta\\ \gamma\end{Bmatrix}=\Sigma g^{\gamma\nu}\begin{bmatrix}\alpha\beta\\ \nu\end{bmatrix} \tag{204}$$

$$\begin{bmatrix}\alpha\beta\\ \nu\end{bmatrix}=\frac{1}{2}\left(g_{\alpha\nu,\,\beta}+g_{\beta\nu,\,\alpha}-g_{\alpha\beta,\,\nu}\right) \tag{205}$$

$$g_{\alpha\beta,\,i}=\frac{\partial g_{\alpha\beta}}{\partial x_i} \tag{206}$$

and $g^{\alpha\beta}$ is the minor of the constituent $g_{\alpha\beta}$ in g divided by g. The quantity C is called the total curvature of the space time. The quantity Λ depends on the gravitational potentials, and their derivatives and on the quantities which define the electromagnetic field and the distribution of matter. Various expressions for Λ which are generalisations of $\frac{1}{2}(B^2-E^2)$ have been suggested but there seems to be some uncertainty as to the exact form.

The calculus of variations leads to the equations

$$2bG_{\alpha\beta}\sqrt{-g}-\lambda g_{\alpha\beta}=C_{\alpha\beta} \tag{207}$$

where $C_{\alpha\beta}$ are the components of the symmetrical tensor of the electromagnetic and mass fields. These are Einstein's gravitational equations.

Many attempts have been made to derive the structure of the electron from Einstein's theory and its generalizations but they do not seem, in the writer's opinion, to be as promising as that sketched in Section III.

In connection with the problem of deriving dynamical equations of motion of the usual type from the analysis of Section II it may be recalled that Lorentz,[4] Abraham,[3] Born[99] and others[*] have derived the dynamical equations of motion of an electron by equating to zero the integral

$$\int\rho\left[E+\frac{1}{c}(v\times H)\right]dxdydz \tag{208}$$

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[*] H. Poincaré, Archives Néerlandaises, t. 5 (1900), p. 252, J. Larmor, l.c.
E. Kohl, Ann. d. Phys. Bd. 13 (1904), p. 770.

taken over the electron.* This is in a sense a generalization of the Principle of d'Alembert. The equations of motion derived from this principle may be expressed in the approximate form†

$$mf-n\dot{f}=e\left\{E_0+\frac{1}{c}(v\times H_0)\right\}$$

where f is the acceleration of the electron, e, the total charge (E_0H_0) the electromagnetic field arising from external charges, a the radius of the electron and

$$m=\frac{e^2}{6\pi ac^2}, \qquad n=\frac{e^2}{6\pi c^3}.$$

In the case of uniform circular motion the term $-nf$ represents the drag due to the electromagnetic radiation.

A natural generalization of this result may perhaps, be obtained by equating to zero the integral

$$\int\left\{\rho\left[E+\frac{1}{c}(v\times H)+2\underline{\Psi}\nabla\left(\rho\sqrt{1-\frac{v^2}{c^2}}\right)\right\}dxdydz$$

taken over the electron. The exact form of the equations of motion has not yet been obtained.

In conclusion it may be remarked that an extension of Section II is needed which will take into account the action of a gravitational field. The generalization of $\rho\sqrt{1-\frac{v^2}{c^2}}$ can be written down immediately. in the form

$$\rho g^{-\frac{1}{2}}\left[\Sigma g_{\alpha\beta}\frac{dx_\alpha}{dt}\frac{dy_\alpha}{dt}\right]^{\frac{1}{2}}$$

but the writer feels uncertain as to the exact form which should be given to the stress-energy tensor.§

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* The principle used by Born is slightly different from this. He reduces the electron to a state in which it is momentarily at rest by using a relativity transformation and integrates the product of the stationary ρ and the stationary force over the stationary form of the electron. This procedure is suggested by the theory of relativity. Another form of the principle is used by E. Fermi, Phys. Zeitschr. (1922), p. 340.

† The analysis is given in full by Leigh Page "An Introduction to Electrodynamics," p. 50.

§ It should be mentioned that another tensor has been found which also leads to the equations (13 and has some advantages over the tensor defined by equations (11). With both tensors there is, however, the drawback that both positive and negative energy are radiated outwards from an accelerated electric pole.

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The law of force given in Section II suggests two possible ways in which the binding forces between the constituents of a nucleus may be diminished. In the first place the density gradients inside each constituent may be changed and the changes may be produced either by the action of charges of the same sign as that of the constituent or by charges of various signs. It is difficult to say which cause will be most effective but in any case Lindemann's theory seems quite plausible. The second possibility is that some of the nuclear electrons may accidentally come close to an aggregate of positive charges in the nucleus and so lower the positive potential as to reduce the binding forces and render the system unstable.

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PART IV

THE TROUTON-NOBLE EXPERIMENT

BY E. H. KENNARD

SYMBOLS

c = velocity of light.
E = electric intensity in electrostatic units.
f = force per unit charge (stationary or moving).
G = electromagnetic momentum, g = same per unit volume.
H = magnetic intensity in electromagnetic units.
K = dielectric constant.
L = torque.
N = number of doublets per unit volume.
P = polarization (displacement of electricity in dielectric matter).
U = electric energy in ether $= \iiint E^2/8\pi \, d\tau$.
u = velocity of convection.
V = electrostatic potential.
v = velocity of motion of electricity.
$\beta = u/c$.
$\gamma = 1/\sqrt{1-\beta^2}$.
θ = angle of rotation of condenser.
ρ = volume density of electricity in electrostatic units.
σ = surface density of electricity in electrostatic units.
φ = retarded scalar potential.
Ψ = convection potential.
$\cdot \times$ denotes scalar and vector products, resp.
∇ denotes gradient.
Vectors are in black-face type.

I. GENESIS OF THE EXPERIMENT

1. *The Trouton-Noble experiment* originated in a suggestion of G. F. FitzGerald's. We quote F. T. Trouton (1902)[1]: "The fundamental idea of the experiment is that a charged condenser, when moving through the ether, with its plates edgeways to the direction of motion, possesses a magnetic field between the plates in consequence of its motion" · · · "The question then naturally arises as to the source supplying the energy required to produce this magnetic field." "FitzGerald's view" · · · "was that it would be found to be supplied through there being a mechanical drag on the condenser itself at the moment of charging, very similar to that which would occur were the mass of any body situated on the surface of the Earth to suddenly become greater."

An impulse due to such a drag was looked for by Trouton, using a condenser mounted on one end of a cross arm, but with a negative result; the experiment is described in the paper just cited. Unfortunately the method employed to test the sensitiveness of the apparatus was dynamically fallacious and the effect was really far too small to be detected[2] (see also below, Sec. 5).

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[1] Scientific writings of G. F. FitzGerald, p. 557.

Having thus convinced himself that the FitzGerald drag did not exist, Trouton came to the conclusion that a charged condenser moving through the ether with its plates inclined to the direction of motion ought to experience a torque tending to turn it so as to diminish its magnetic energy and therefore so as to set its plates perpendicular to the motion. Modern theory confirms this conclusion except that the direction of the torque is just the opposite of that described.

This effect was sought in the Trouton-Noble experiment.

II. THE EXPERIMENT

2. *Description.*[1] A mica-plate condenser 7.7 cm. in diameter and having a capacity of 0.0037 microfarads was suspended with its plates vertical from a phosphor-bronze strip 37 cm. long. The condenser was charged to 2100 volts by way of the suspension and a separate wire on the bottom dipping into dilute sulphuric acid. Deflections about the vertical axis were observed upon charging the condenser, by means of a mirror and telescope.

The sensitiveness of the apparatus was determined by finding the elastic constant of the suspension in a separate vibration experiment. The torque that ought to be exerted upon the condenser by the field was then calculated upon the assumption that the torque is equal to the angular rate of decrease of the magnetic energy.

The principal observations were taken during ten days in March, at which time the horizontal drift through the ether was shown to be near a maximum if one included the supposed velocity of the solar system through the ether; three of the observations were taken at noon, when the effects of the earth's orbital motion considered by itself would be a maximum, the others at 3 or 6 P.M. The plane of the plates lay N.E.–S.W.

Eleven observations are reported. The calculated deflections as given by the authors ranged from +.8 to −2.6 cm. at 1 M. distance for the orbital motion alone, and from 0 to −6.8 cm. for the combined motion. The observed deflections ranged from −.12 to −.35 cm., without obvious correlation with other circumstances.

3. *Critique.* A study of the report leaves one with the conviction that the experiment was ably performed and that the data given are reliable. But two points in the interpretation are, at least nowadays, open to serious criticism.

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[2] H. A. Lorentz, K. Akad. Amst. Proc. 6, p. 830, 1904.

[1] Trouton and Noble, Phil. Trans. A, 202, p. 165, 1903.

It is regrettable that good observations were not made and reported for several different periods distributed throughout the year. It is certainly possible that the earth might be almost stationary in the ether during March in consequence of the sun's proper motion through the ether, which we have no really good means of estimating. Or, again, the earth's speed might have been considerable during the period of the observations and yet, as an inspection of the geometrical relationships shows, its component of velocity in a suitable horizontal direction, which would alone be effective in producing a twist about the vertical, might have been always comparatively small. Probably our conviction as to the correctness of the result will not be greatly shaken by this circumstance, especially since the final results were obtained only "after many months of experience" with the apparatus; nevertheless, in contrast, the Michelson-Morley experiment certainly gains considerably in conclusiveness through the avoidance of just this sort of doubt.

A more serious matter is that the authors omitted the "Roentgen current," hardly known in their day, in calculating the magnetic field in the condenser. For they put $H=4\pi\sigma w$ where w = velocity parallel to the plates and σ = surface density of charge on the conducting plates (in electromagnetic units). But the convection of the apparent charges on the mica dielectric also contributes to the magnetic field, and in the opposite sense, making the resultant field $4\pi\sigma w/K$. This illustrates the fact that in a moving electrostatic system the magnetic intensity due to convection is proportional at every point to the electric intensity.

In consequence the authors overestimate the torque to be expected in the ratio K^2. But later they make a slip by substituting for μK in the numerator v^2 in the denominator, where v = "velocity of electric propagation" in the dielectric, and then taking $v=3.10^{10}$ in their numerical calculation. This wrongly divides the deflection by K and leaves it only K times too great. Thus, taking $K=6$, the deflection to be expected becomes, for the orbital motion, from $+.13$ to $-.43$ cm., or almost within the range of the errors of observation.

III. Theory of a Moving Condenser

The theory of the experiment is presented in detail in Laue's "Relativitätstheorie," but he does not consider the effect of a material dielectric and he supposes the moving condenser to undergo the relativity contraction. The introduction of this contraction does not appreciably alter the result but it seems preferable to omit it in the present discussion since the object of our inquiry is the result predicted by the fixed-ether theory. Accordingly we shall omit the contraction and give an outline of the argument.

4. According to the *Maxwell-Lorentz* theory the fundamental equation for the calculation of all ponderomotive forces of electromagnetic origin is

$$\mathbf{f}=\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{H} \tag{1}$$

where $\mathbf{v}$ denotes velocity of the electricity relative to the ether. The field vectors are connected in turn with the electricity by the equations (the units being ordinary Gaussian)

$$\left.\begin{aligned} \mathit{curl}\,\mathbf{H}&=\frac{1}{c}\left(4\pi\rho\mathbf{v}+\frac{\partial\mathbf{E}}{\partial t}\right). & \mathit{div}\,\mathbf{H}&=0. \\ \mathit{curl}\,\mathbf{E}&=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}. & \mathit{div}\,\mathbf{E}&=4\pi\rho. \end{aligned}\right\} \tag{2}$$

These equations lead, by a purely mathematical calculation, to the following well-known principle:

(A) The moment about any fixed axis of all the forces exerted by the field upon electricity equals the rate of decrease in the total moment about that axis of the electromagnetic momentum in the field, the electromagnetic momentum per unit volume being defined as

$$\mathbf{g}=\frac{1}{4\pi c}\mathbf{E}\times\mathbf{H}. \tag{3}$$

A special case (axis at infinity) is this:

(A') The vector sum of the forces exerted by the field upon electricity equals the rate of decrease of the total vector electromagnetic momentum in the field.

For *convection of an electrostatic system* we easily find further that

$$\mathbf{H}=\frac{1}{c}\mathbf{u}\times\mathbf{E} \tag{4}$$

where $\mathbf{u}$=velocity of convection: while the problem of finding $\mathbf{E}$ can be reduced to an electrostatic problem by the following device:

(B) Imagine the moving system S, including its distribution of electricity, stretched uniformly in the direction of motion in the ratio γ and then brought to rest, forming what we shall call the "associated stationary" system S_1. Then the plot of the lines of $\mathbf{E}$ simply stretches into the plot of lines in S_1, and, letting $\mathbf{E}_1$ and V denote electrostatic intensity and potential, respectively, in S_1 and taking the x-axis parallel to the direction of motion, we have the following relationships between magnitudes at corresponding points in the two systems:

$$\varphi=\gamma V,\quad E_x=E_{1x},\quad E_{y,z}=\gamma E_{1y,z} \tag{5}$$

where φ=retarded scalar potential in S.

From this we find easily that

$$\mathbf{f} = -\nabla\psi,\ \psi = (1-\beta^2)\varphi = V/\gamma, \tag{6}$$

$$f_x = E_x = E_{1x},\ f_{y,z} = (1-\beta^2)E_{y,z} = 1/\gamma\ E_{1y,z} \tag{7}$$

where ψ is the "convection potential" and $\mathbf{f}$, the force per unit charge.

Clearly the condition for electric equilibrium of a conductor is that in S ψ must be uniform, and in S_1, V; but (6) shows that both conditions are of necessity met simultaneously. Accordingly:

(*C*) To find the distribution of given charges which will be in equilibrium on a set of conductors in S, we have only to find the distribution of the same charges in electrostatic equilibrium in S_1 and then contract everything in the ratio $1/\gamma$ parallel to the motion.

Proofs of these principles and formulas may be found in Abraham's *Theorie der Elektrizität*, Vol. II, or in Richardson's *Electron Theory of Matter*.

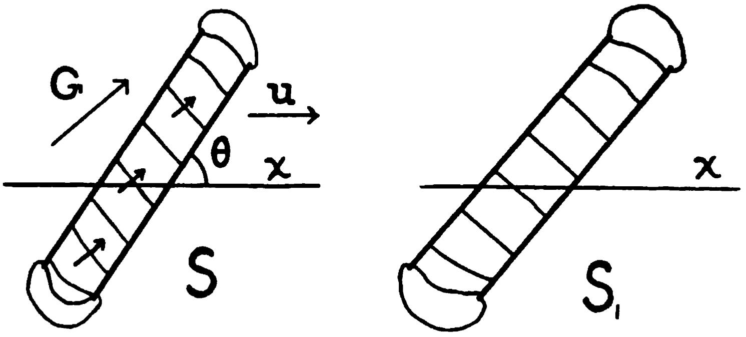

FIG. 1.

5. *Condenser with Ether as Dielectric*: *β small.*—Applying (B), we have as S_1 a stationary condenser that is slightly warped but with the plates still parallel. The warping slightly distorts the field in S_1; and then when we contract it to obtain the electric field in S, the lines become slightly inclined to the plates as shown. Accordingly, (3) and (4) show that $\mathbf{g}$ is inclined to the plates at a small angle (as shown by the arrows in the figure). But since the contraction is of order β^2 we shall incur relative errors only of the same order if we assume that the electric intensity is the same in the moving condenser as if there were no motion. $\mathbf{g}$ is then practically parallel to the plates and (3) and (4) give for its magnitude

$$g = \frac{\beta}{4\pi c} E^2 \cos\theta \tag{8}$$

where θ = angle between the plates and the direction of motion. The total momentum is therefore

$$G=\frac{\beta\cos\theta}{4\pi c}\iiint E^2 d\tau \text{ or } G=\frac{2\beta}{c}U\cos\theta, \tag{9}$$

where U = electric energy in the ether, the direction of **G** being practically parallel to the plates as shown in the figure.

A further slight error is introduced by the edge-field but this can be reduced indefinitely in the usual way by approximating the plates. The calculation given in Laue differs by a further small quantity of order β^2 because there the system S is supposed to undergo the Relativity contraction.

The *mechanical reactions* of the field upon the condenser are now readily obtained.

When the condenser is charged, (A′) shows that it will experience a backward impulse (the FitzGerald drag) nearly parallel to the plates and equal to $(2\beta/c)\ U\cos\theta$. For the earth's orbital motion $\beta=10^{-4}$, so that the maximum impulse would be less than $U\cdot 10^{-14}$. With modern apparatus a linear impulsive velocity of 10^{-3} cm. per sec. ought to be measurable to 10 per cent; but then $10^{-3}m=U\cdot 10^{-14}$, $U/m=10^{11}$, or we should have to load the condenser with 10^4 joules of electrical energy per gram of weight. At present this is probably impossible.

A more important conclusion for our present purpose is that the condenser also experiences a torque. For, taking a fixed axis at right angles to **G** and to the direction of motion (normal to the paper in the figure), we see that, since **G** is being carried along with a transverse component of velocity $u\sin\theta$, the moment of **G** about this axis is steadily increasing at the rate $Gu\sin\theta$ or, by (9), $2\beta^2U\sin\theta\cos\theta$; hence by principle (A) there is a torque on the condenser equal to

$$L=2\beta^2U\sin\theta\cos\theta; \tag{10}$$

and its direction will clearly be such (clockwise in fig.) as to tend to set the plates parallel to the motion.

This result confirms Trouton and Noble's calculation for the case $K=1$ except that the direction of the torque is reversed. The torque is small but ought to be readily detectable.

As a check, we shall attempt to calculate the torque directly from (1) and (4). At any point between the plates $H=\beta E\cos\theta$; as we pass through the charged layer on one plate the magnetic intensity decreases in proportion to the amount of charge passed over and becomes zero at a point outside of the charged layer. Hence one easily finds for the force per unit area on the plate $(H/2)(\sigma u/c)$ or, since $\sigma=E/4\pi$, $(\beta^2E^2$

cos $\theta)/8\pi$, the force being perpendicular to the direction of motion. Pairing off opposite elements of the two plates, we find a torque upon them equal per unit area to this force times the distance between the plates times sin θ, and summing for the whole condenser we arrive at just half of L as given by (10).

The discrepancy is not caused by the electric intensity, for this can be seen to exert no torque if we resolve the field into the elementary fields due to the separate elements of charge. In each of the latter fields the electric intensity points away from the element producing it and the law of action and reaction consequently holds for the interaction due to this cause between the members of any pair of elements.

As a matter of fact, the discrepancy is due to an excess force on the edges of the plate whose moment increases with the size of the plates and hence cannot be made negligible by increasing indefinitely the ratio of area to plate distance; details of this, slightly modified by the relativity contraction, are given in Laue's *Relativitätstheorie.*

6. *Effect of a Material Dielectric of Unit Permeability.*—This deserves careful consideration because mica condensers were used in the experiment.

Part of the theory can be extended to this case simply by including the polarization electricity in our distribution of charge. For then principles (A), (A′), (B), and equations (1)–(2), (4)–(7) will still hold and not only for the values at a point but also for the observable values of the vectors and the density, which represent mean values taken throughout a "physically small" volume. But (3), which is not linear, and (C) require elaboration.

The polarization **P** in S, being measurable as electricity displaced across unit area, stretches into a polarization $\mathbf{P}_1$ in S_1 given by

$$P_x = P_{1x}, \quad P_{y,z} = \gamma P_{1y,z} \tag{11}$$

Now if the dielectric were isotropic in both systems **P** would be parallel to **f** and $\mathbf{P}_1$ to $\mathbf{E}_1$; but then (11) would be inconsistent with (7). Hence we shall suppose an isotropic dielectric in S to become anisotropic in S_1, the polarization constant, $(K\text{-}1)/4\pi$, being decreased in a direction transverse to the motion in the ratio $1/\gamma^2$, so that

$$K_{1x} = K, \quad K_{1y,z} = K - \beta^2(K-1) \tag{12}$$

The polarization will then be in equilibrium in both systems simultaneously, and it follows that principle (C) can be extended to this case. The slight distortion of the field produced by the anisotropy in S_1 is easily seen to be of no appreciable consequence in the present connection.

Finally, in (3) let us split **E** and **H** into two parts and write $\mathbf{E} = \mathbf{E}' + \mathbf{E}''$, $\mathbf{H} = \mathbf{H}' + \mathbf{H}''$, where the plain letters denote values at an actual point and the single-primed letters, the observed mean values; then on the average $\mathbf{E}'\times\mathbf{H}''=\mathbf{E}''\times\mathbf{H}'=0$, and the mean momentum per unit volume is

$$\mathbf{g}'=\frac{1}{4\pi c}\left[\mathbf{E}'\times\mathbf{H}'+\mathbf{E}''\times\mathbf{H}''\right]. \tag{13}$$

The first term on the right leads at once to equations (8)–(10), that is, this term gives, for a fixed potential difference between the plates, the same value for the torque as if the material dielectric were removed. This is the result to which we are led if we follow the usual rule of simply ignoring fine-grained irregularities of field.

7. *Local Torque in a Dielectric.* The second term on the right in (13) requires special consideration. The general result appears to be that it yields an additional torque which may greatly exceed the one given by (10) but which can have either sign and which has nothing to do with the ordinary electrical properties of the substance but depends directly upon its atomic structure. Indeed, one would expect a torque to exist in many crystals even in the absence of an electric field.

We shall illustrate the possibilities of the case by considering a simple group of electric doublets whose dimensions are small relative to their distance apart.

The mechanical forces upon a given doublet arise in part from the field of its neighbors and of distant charges. This part will depend only upon the strengths and positions of the doublets and of other charges.

A second part arises from the magnetic interaction between the constituent charges of each doublet. If the latter are $+e$ and $-e$ and are separated by a displacement l making an angle ζ with the x-axis, which we suppose to be the direction of motion, the doublet experiences a torque tending to set its axis at right-angles to the direction of motion and of magnitude (the magnetic intensity being $\beta e \sin\zeta/l^2$).

$$L_1=\frac{\beta^2 e^2}{l}\sin\zeta\cos\zeta. \tag{14}$$

Let there be N doublets per unit volume with parallel axes. Then the torque per unit volume is $L=NL_1$, while the resulting polarization is $P=Nel$, so that the torque per unit volume can also be written

$$L=\frac{\beta^2 P^2}{Nl^3}\sin\zeta\cos\zeta. \tag{15}$$

This torque can attain any magnitude independently of P through variation in l. Hence a counterbalancing of it by other effects which depend upon P can occur only as a matter of accident. The denominator Nl^3 is, according to our assumptions, much less than unity.

From this result one seems justified in drawing the broad conclusion that *crystals* ought, in general, to experience an appreciable torque when moving through the ether. Definite calculations cannot be made without the adoption of some definite atomic theory, but all modern theories suppose electrical separations to exist in the atom which by themselves would produce enormous values of P, and in general one would expect these to give rise to an outstanding torque of the magnitude of L as given by (15) with P a large number and Nl^3 at least less than unity. For instance, if 10^{23} electrons per unit volume were displaced relative to the atoms a distance of only 10^{-10} cm. the resulting polarization would be $10^{23} \cdot 10^{-10} \times 4.77 \cdot 10^{-10} = 4{,}770$ electrostatic units. This value of P gives, even with Nl^3 replaced by unity, a value of L which is hundreds of times bigger than the torque per unit volume on Trouton and Noble's condenser; for in the latter case the electric intensity appears not to have exceeded 700 electrostatic units, so that U in (10) divided by the volume would be about 20,000.

Since the occurrence of such a torque would constitute an effect of uniform motion through the ether, Relativity requires that any torque due to this cause must be compensated within the crystal by an equal and opposite torque of different origin (presumably of the same nature as that described below in Sec. 10). Thus the fact that no such torque has ever been observed lends a certain amount of support to Relativity.

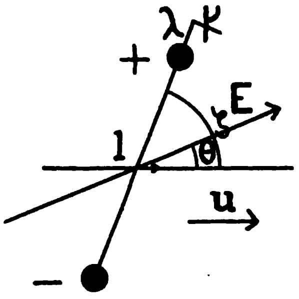

FIG. 2.

To illustrate now the possibilities in an *isotropic* medium, let us consider the effect of applying an electric field in a direction inclined at an angle θ to the direction of motion (Fig. 2), and let us suppose the doublets to be oriented irregularly except that their axes are all parallel to the plane defined by these two directions. Let the field displace the positive charge a small distance λ in the direction of the axis and a small distance μ at right angles to it where

$$\lambda = aE \cos(\zeta - \theta), \quad \mu = bE \sin(\zeta - \theta). \tag{16}$$

In this case there is an outstanding second-order torque for which, using (14), I find the value

$$L=\frac{N\beta^2e^2}{16l^3}(2a^2-8ab+5b^2)E^2\ sin\ 2\theta,$$

or, introducing $P=Ne(a+b)E/2=(K-1)E/4\pi$,

$$L=\left[\frac{(K-1)^2}{8\pi Nl^3}\ \frac{2a^2-8ab+5b^2}{(a+b)^2}\right]\frac{\beta^2E^2}{8\pi}\ sin\ 2\theta, \tag{17}$$

where L = torque per unit volume.

This expression is the same as (10) divided throughout by the volume of the condenser, except for the factor in brackets. It is hard to assign a plausible value to the latter factor without assuming some definite theory of atomic structure; but according to our assumptions Nl^3 is small compared with unity, hence the whole bracket ought for a solid dielectric to be comparable with unity and might greatly exceed this value, and it might have either sign. For instance, for $Nl^3=1$, $K=6$ (mica) and $a=0$ (approximately pure rotation of the doublets) the bracket equals $+5$, while for $Nl^3=1/10$, $K=6$ and $a=1.5\ b$ the bracket equals -4.

In an actual dielectric, circumstances will no doubt be very different from those of the simple case here treated. Yet we seem justified in concluding that, according to the electromagnetics of Lorentz, a local torque of appreciable magnitude is very likely to act upon a moving polarized dielectric and that this torque might conceivably mask completely the effects of the torques upon the true and apparent charges.

8. *The Effect of the Mica in Trouton and Noble's Condenser* remains therefore in doubt. It may have caused the null result; more likely, however, being crystalline, it should have greatly increased the effect, perhaps with a reversal of sign.

9. *The Fine-Structure of the Charge on the Plates* might conceivably have a similar effect. The removal or addition of electrons on the surface might form doublets with axes more or less parallel to the surface, and these, by urging the plates toward a position at right angles to the motion, might mask the main effect. Such an effect could be distinguished in a repetition of the experiment through the circumstance that it would depend only on the charge and not, like the main torque, also upon the difference of potential.

10. *The Explanation by Relativity* of the null result is a dynamical one and is fully given in Laue's *Relativitätstheorie*. The torque occurs as a secondary effect superposed upon the far larger electrostatic attraction between the plates: according to Relativity, the intermolecular stresses which balance the latter attraction do not obey Newton's

Third Law but themselves produce a torque which just balances the electromagnetic one. The null result is thus explained, not exactly by the Lorentz-FitzGerald contraction itself, but rather as a consequence of the same cause that produces this contraction.

11. *Other Ways of Excape* are hard to find. So far as the writer is aware, there is no rival to the Maxwell-Lorentz theory which explains all ordinary phenomena (including Hertzian waves) and also removes the torque on the condenser. Of course, drag of the ether by the earth would do it, but this assumption leads to well-known difficulties. Instead of speculating upon possible modifications of electromagnetic theory it seems more profitable to pass in review the experimental facts upon which the prediction of the torque rests. They are (for vacuum as dielectric):

(1). Moving charged bodies generate a magnetic field. The charged body used in experiments like Rowland's is very similar to either plate of our condenser, the only important difference being that in those experiments the average convection current was closed.

(2). Magnetic fields (of certain kinds, at least) act on moving charged bodies (probably verified only for charged molecules or electrons).

(3). No difference has yet been detected between magnetic fields arising from different causes.

These basic facts lead by very simple reasoning to the predicted torque. As a matter of logic one might attack the sufficiency of the present experimental basis for (2) and (3), but the prospect of discovering any error at this point seems very small. Perhaps the most promising thing to try out would be whether a moving charged *conductor* really is acted upon by a magnetic field.

12. *In Conclusion*, the situation may be summed up as follows:

Trouton and Noble's negative result might have been due to either (1) insufficient sensitiveness of their apparatus (Sec. 5), or (2) insufficient distribution of their observations over different times of year (Sec. 5), or (3) a special effect of the mica dielectric (Sec. 7, 8), or (4) a similar special effect in the charged surface of the plates (Sec. 7, 9).

If, however, it be accepted as an experimental fact that an air condenser never experiences a torque due to the earth's motion, then this fact speaks forcibly in favor of Relativity and can hardly be explained on any other basis.

Probably few physicists will refuse today to accept this as an experimental fact, nevertheless the importance of the problem seems to justify a repetition of the experiment, with observations taken on an air condenser, at different times both of day and of year, and with various distances between the plates.

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